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Analysis of 2-Vertex Self-Switching in Two-Cyclic Graphs: Connected and Disconnected Cases

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Abstract: When discussing a finite undirected graph, $G(V, E)$ and a subset that is not empty $\sigma \subseteq V$, the graph created by switching of G by σ is expressed as $G^\sigma(V, E')$. This particular graph is derived from G by all non-edges between σ and $V - \sigma$ are added together and cutting off every edge between σ and its counterpart, $V - \sigma$. For $\sigma = \{v\}$, we record G^v, and this transformation is alternatively known as vertex switching. However, this is what we call $|\sigma|$-vertex switching. If $|\sigma| = 2$, it is presented as 2 -vertex switching. In a two-cyclic graph, there are precisely two cycles. More than one component makes up a disconnected graph. There are no isolated components in a connected graph. We describe two-vertex self switching of two-cyclic graphs in this paper.

Keywords: *Switching, Connected two cyclic graphs, Disconnected two cyclic graphs, 2 -vertex self switching, 2-vertex Switching.*
Subject Classification: 05C40, 05C07 05C40.

1 Introduction

If $V(G) = p$ is present in any graph $R(V, E)$, then $R'(V, E')$ is the graph that is created from R by pulling out every edges between σ and its counterpart, $V - \sigma$, and also every non-edges between σ and $V - \sigma$ are added together as edges, in which $\sigma \subseteq V$. Switching was interpreted by Seidel [1, 6], alternatively mentioned as $|\sigma|$-vertex switching. In the scenario where $|\sigma| = 2$, it is named as 2 -vertex switching. The branches and joints in graphs were first conceptualised by Vilfred V et al.,[8]. In G, a joint at σ is a subgraph B of G which contains $G[\sigma]$ if $B - \sigma$ is maximal and connected. We call it a c-joint if B is connected, and a d-joint otherwise. $B = G[\sigma] + (B - \sigma)$ indicates that B is a total joint. A graph is referred to as unicylic if it has precisely one cycle. In [3], the notion of 2-vertex switchings of unicyclic graphs was examined. Two-cyclic graphs are those that have precisely two cycles. In [4], The idea of self-vertex switchings of two-cyclic graphs was examined.

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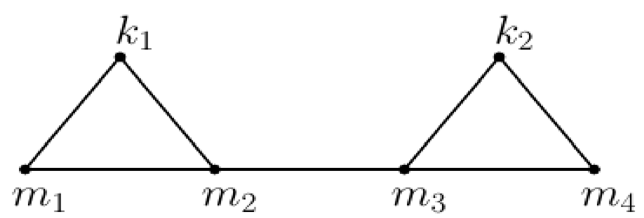
Alternative triangular snake graphs were explored in [7]. $A(T_n)$ is an alternate triangular snake which is constructed from a path $m_1, m_2, \ldots, m_n$ by connecting each m_i (alternatingly)with a new vertex k_i, which is then joined to m_i and m_{i+1} through k_i. That is, C_3 replaces each alternate edge of a path. The alternative triangular snake graph with $n = 4$ is given in figure 1.1



$Fig1.1 : A(T_4)$

This study offers a comprehensive description of two-cyclic graphs that possess a 2 -vertex self switching. Our study also includes graphs that feature c-joints and d-joints of 2 - vertex switching. Throughout this paper, we used by $d_G(u)$ the degree of the vertex u in the graph G.

We now present the following theorems, which serve as foundational results for the next section.

Theorem 1.1. [5] Let G be a graph of order $p \geq 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that uv $\notin E(G)$. Let B be a d-joint at σ in G. Then B^σ is a c-joint at σ in G^σ if and only if $B - \sigma$ is connected and either $d_B(u) = 0$ and $0 \leq d_B(v) \leq |V(B)| - 3$ or $d_B(v) = 0$ and $0 \leq d_B(u) \leq |V(B)| - 3$.

Theorem 1.2. [5] Let G be a graph of order $p \geq 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that uv $\notin E(G)$. Let B be a d-joint at σ in G. Then B^σ is a d-joint at σ in G^σ if and only if $B - \sigma$ is connected and $\{d_B(u), d_B(v)\} = \{0, |V(B)| - 2\}$.

Theorem 1.3. [5] Let G be a graph of order $p \geq 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \in E(G)$. Then B is a d-joint at σ in G if and only if B^σ is a total joint at σ in G^σ.

Corollary 1.4. [9] Let G be a graph of order $p \geq 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \in E(G)$. If B is a d-joint at σ in G, then G^σ cannot be a d-joint.

Theorem 1.5. [5] Let G be a graph of order $p \geq 3$ and let $\sigma = \{u, v\}$ be a subset of $V(G)$ such that $uv \in E(G)$. Let B be a c-joint at σ in G. Then B^σ is a d-joint at σ in G^σ if and only if $B - \sigma$ is connected and $d_B(u) = d_B(v) = |V(B)| - 1$.

Theorem 1.6. [2] Let G be a graph of order $p \geq 5$ and let $\sigma = \{u, v\} \subseteq V(G)$ be such that uv $\notin E(G)$. Let M be the set of non-adjacent vertices of u and N be the set of non-adjacent vertices of v in G. Let G be a c-joint at σ. Then G^σ is a c-joint and two-cyclic at σ if and only if $|V(G)| \geq 5$ and one of the following holds:

1. $G - \sigma$ is connected, acyclic, $d_G(u) = d_G(v) = |V(G)| - 4$ and the path formed by elements of M and by elements of N have either at most one vertex in common for $M \cap N = \varphi$ or the vertex a in common for $M \cap N = \{a\}$.
2. $G - \sigma$ is connected, unicyclic, $\{d_G(u), d_G(v)\} = \{|V(G)| - 4, |V(G)| - 3\}$ and for $|M - \{v\}| = 2$ and $|N - \{u\}| = 1(|M - \{v\}| = 1$ and $|N - \{u\}| = 2)$ either the elements of $M(N)$ do not lie on the cycle of $G - \sigma$ and the unique path connecting them contains at most one vertex of the cycle or one of the elements of $M(N)$, say a, lies on the cycle and the unique path connecting them contains no vertex of the cycle other than a.
3. $G - \sigma$ is connected, two-cyclic and $d_G(u) = d_G(v) = |V(G)| - 3$.

2 Core Findings

2-vertex Self-Switching of Connected and Disconnected Two-Cyclic Graphs

Theorem 2.1. Assume G is a graph for which $|V(G)| = p$ with $= \{u, v\} \subseteq V(G)$. Define M as the set of nodes in G that are not neighbors of u, and N similarly for v. If B is a d-joint in G, then B^σ cannot be two-cyclic.

Proof. Suppose B^σ is a two-cyclic graph. Let M and N represent the set of non-adjacent vertices of u and v, respectively in G.

Since B^σ is two-cyclic, B has at least 5 vertices. Since $|V(B)| \geq 5, |\, V(B) - \ \sigma \,| \geq 5 - 2 \geq 3$. We have two cases based on $uv \notin E(G)$ and $uv \in E(G)$.

Case1.$uv \notin E(G)$
Subcase 1.1. The graph B is a d-joint, while B^σ is a c-joint
According to Theorem 1.1, $B - \sigma$ is connected. Also, either $d_B(u) = 0$ with $d_B(v)$ satisfying $0 \leq d_B(v) \leq |V(B)| - 3$, or $d_B(v) = 0$ with $d_B(u)$ satisfying $0 \leq d_B(u) \leq |V(B)| - 3$. Without loss of generality, choose $d_B(u) = 0$. Therefore, every elements of M in $V(B) - \sigma$ is neighbor of u in B^σ. Since $|V(B) - \sigma| \geq 3, M$ has at least 3 nodes, say α, κ and δ of $V(B) - \sigma$ for which α, κ and δ are neighbor of u in B^σ. There are $\alpha - \kappa, \kappa - \delta$, and $\alpha - \delta$ paths in B^σ, Since $B - \sigma$ is connected. Hence, the edges $u\alpha, u\kappa$ and $u\delta$, the paths $\alpha - \kappa, \kappa - \delta$, and $\alpha - \delta$ form at least three cycles in B^σ which contradicts our assumption that B^σ is two cyclic.

Subcase 1.2. The graph B is a d-joint, while B^{σ} is a d-joint

According to Theorem 1.2, $B-\sigma$ is connected and $\{d_B(u), d_B(v)\} = \{0, |V(B)|-2\}$. We take $d_B(u)=0$ and $d_B(v)=|V(B)|-2$ to maintain genarality. It then follows, by an argument analogous to that in Subcase 1.1, we get B^{σ} has at least three cycles which contradicts B^{σ} is two cyclic.

Case2. $uv \in E(G)$

Subcase 2.1. The graph B is a d-joint, while B^{σ} is a c-joint

By Theorem 1.3, we have B^{σ} is a total joint at σ in G^{σ}. Hence, $d_{B^{\sigma}}(u) = d_{B^{\sigma}}(v) = |V(B)|-1$. Since $uv \in E(G)$, uv is a component in B. Since $|V(B)-\sigma| \geq 3, M$ has at least three vertices, say β, λ and Γ, such that β, λ and Γ are adjacent to u in B^{σ}. Hence, the edges $uv, u\beta, u\lambda, u\Gamma$ and the paths $\beta-\lambda, \lambda-\Gamma, \beta-\Gamma$ form more than two cycles in B^{σ} which contradicts B^{σ} is two cyclic.

Subcase 2.2. The graph B is a d-joint, while B^{σ} is a d-joint

By Corollory 1.4, B^{σ} cannot be a d-joint. Hence this case does not exist.
From the above cases, we conclude that B^{σ} is not a two cyclic graph. Hence, we obtain the desired result.

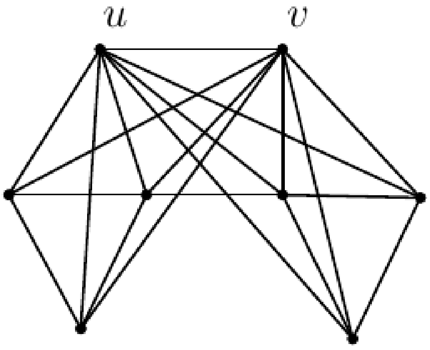
Theorem 2.2. Consider a graph G such that $|V(G)| = p \geq 5$ with $\sigma = \{u, v\} \subseteq V(G)$ for which $uv \in E(G)$. Define M as the set of nodes in G that are not neighbors of u, and N similarly for v. If B is a c-joint at σ in G, it follows that B^{σ} is a d-joint and two-cyclic at σ in G^{σ} if and only if $B-\sigma$ is connected, two-cyclic with $d_B(u) = d_B(v) =| V(B)-1$.

Proof. Assume B be a c-joint at σ in G and B^{σ} is a d-joint and two-cyclic at σ in G^{σ}. By Theorem 1.5, we have $d_B(u) = d_B(v) =| V(B)-1$ and $B-\sigma$ is connected. Since $uv \in E(G)$ and $d_B(u) = d_B(v) = | V(B)-1, B$ is a total joint at σ in G. Hence, $B^{\sigma} = K_2 \cup (B-\sigma)$ and so $B-\sigma$ is two-cyclic as B^{σ} is two-cyclic.

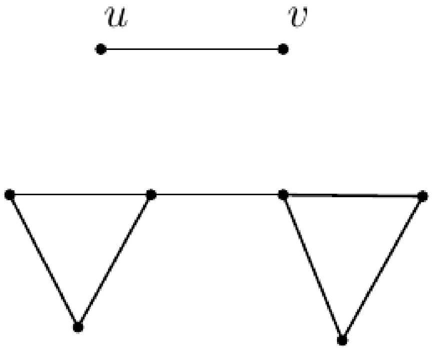
Conversely, Assume $B-\sigma$ is connected, two-cyclic and $d_B(u) = d_B(v) = |V(B)|-1$. According to Theorem 1.5, B^{σ} is a d-joint at σ in G^{σ}. Since B^{σ} is d-joint with $d_B(u) = d_B(v) =| V(B)-1, B^{\sigma} = K_2 \cup (B-\sigma)$, where K_2 is the edge of uv. Since $B-\sigma$ is two-cyclic, B^{σ} is also two-cyclic. Hence, we obtain the desired result.

Corollary 2.3. Consider a graph G such that $|V(G)| = p \geq 5$ with $\sigma = \{u, v\} \subseteq V(G)$ such that $uv \in E(G)$. Define M as the set of nodes in G that are not neighbors of u, and N similarly for v. If G is a c-joint at σ, it follows that G^{σ} is a d-joint and two-cyclic at σ if and only if $G-\sigma$ is connected, two-cyclic with $d_G(u) = d_G(v) = p-1$.

Example 2.4. Assume the graph G, having 8 vertices, as depicted in figure 2.1. In this graph, G is a c-joint at the vertex set $\sigma = \{u, v\}$, where the degree of the vertices satisfy $d_G(u) = d_G(v) = |V(G)|-1 = 7$. The graph G^{σ} is illustrated in figure 2.2 is a d-joint at σ. Clearly, G^{σ} is a disconnected and two-cyclic graph.



$Fig\ 2.1 :\ G$



$Fig\ 2.2 :\ G^{\sigma}$

Theorem 2.5. Assume G is a connected graph with $|V(G)| = p \geq 5$ containing exactly two cycles for which $\sigma = \{u, v\} \subseteq V(G)$, where $uv \notin E(G)$ and $G - \sigma$ is connceted. Then G has a 2 -vetex self switching at σ if and only if G is either $A(T_4), C_{3(u)}[0, C_4, 0]$ or $C_{3(u)}[0,0, C_4]$ with $d_G(u) = d_G(v) = 2$.

Proof. Let G denote a connected graph containing exactly two cycles. Let $\sigma = \{u, v\} \subseteq V(G)$ be a 2-vertex self switching of graph G. Then $G \cong G^\sigma$ and so G^σ is connceted and two-cyclic.

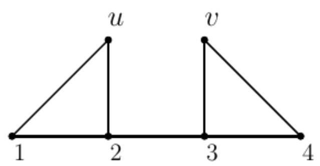
Let M and N represent the set of non-adjacent vertices of u and v, respectively in G. Since $uv \notin E(G)$, by Theorem 1.6, $|V(G)| \geq 5$ and either $G - \sigma$ is connected, acyclic, $d_G(u) = d_G(v) = |V(G)| - 4$ and the path formed by elements of M and by elements of N have either at most one vertex in common for $M \cap N = \varphi$ or the vertex a in common for $M \cap N = \{a\}$ or $G - \sigma$ is connected, unicyclic, $\{d_G(u), d_G(v)\} = \{|V(G)| - 4, |V(G)| - 3\}$ and for $|M - \{v\}| = 2$ and $|N - \{u\}| = 1(|M - \{v\}| = 1$ and $|N - \{u\}| = 2)$ either the elements of $M(N)$ do not lie on the cycle of $G - \sigma$ and the unique path between them intersects the cycle in no more than one vertex or one of the elements of $M(N)$, say a, lies on the cycle and the unique path between them intersects the cycle only at vertex a or $G - \sigma$ is connected, two-cyclic and $d_G(u) = d_G(v) = |V(G)| - 3$.

Case 1. $G - \sigma$ is connected, acyclic, $d_G(u) = d_G(v) = |V(G)| - 4$ and the path formed by elements of M and by elements of N have either at most one vertex in common for $M \cap N = \varphi$ or the vertex a in common for $M \cap N = \{a\}$

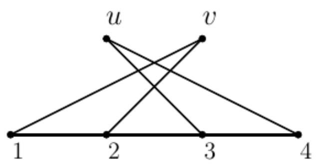
If $|V(G)| = 5$, then $|V(G) - \sigma| = 5 - 2 = 3. G - \sigma = P_3$, Since $G - \sigma$ is acyclic and connected. Moreover $d_G(u) = d_G(v) = 1$ refers that G is either P_5 or $P_3(0,2P_2, 0)$ or $P_3(2P_2, 0,0)$ or $P_3(P_2, P_2, 0)$. This leads to a contradiction,
since G is assumed to be two-cyclic. Therefore, it must be $|V(G)| \geq 6$.

If $|V(G)| \geq 7$, then $d_G(u) = d_G(v) \geq 3$ and $|V(G) - \sigma| \geq 5$. Therefore there exist minimum three vertices, say, γ, Λ and Δ in $G - \sigma$ implies u is neighbor of γ, Λ and Δ. Since $G - \sigma$ is connected, the edges $u\gamma, u\Lambda$ and $u\Delta$ and the paths $\gamma - \Lambda, -\Delta$ and $\gamma - \Delta$ yielding at least three distinct cycles in G, contradiction to G being two-cyclic.

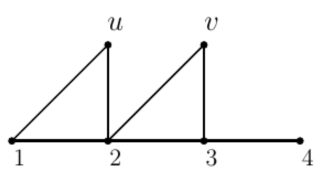
Therefore, $|V(G)| = 6$. This implies that $d_G(u) = d_G(v) = 6 - 4 = 2$ and $|V(G) - \sigma| = 6 - 2 = 4$. Since $G - \sigma$ is connected and acyclic, $G - \sigma$ is either P_4 or $K_{1,3}$. The possible two-cyclic graphs that are not-isomorphic on 6 vertices with $d_G(u) = d_G(v) = 2$ are $A(T_4), C_{3(v)}[0, P_2, C_3], C_{3(u)}[0, C_4, 0]$, $C_{3(u)}[0,0, C_4], C_{3(u)}[0,0, P_2 \cup C_3]$ which are given in figures 2.3,2.5,2.7,2.9 and 2.11.



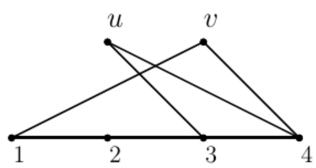
$Fig\ 2.3 : A(T_4)$



$Fig\ 2.4 : A(T_4)^\sigma$



$Fig\ 2.5 : C_{3(v)}[0, P_2, C_3]$



$Fig\ 2.6 : C^\sigma_{3(v)}[0, P_2, C_3]$

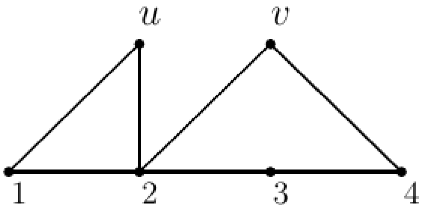


Fig 2.7 : $C_{3(u)}[0, C_4, 0]$

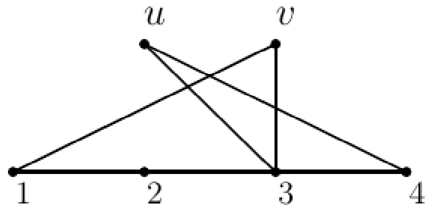


Fig 2.8 : $C^{\sigma}_{3(u)}[0, C_4, 0]$

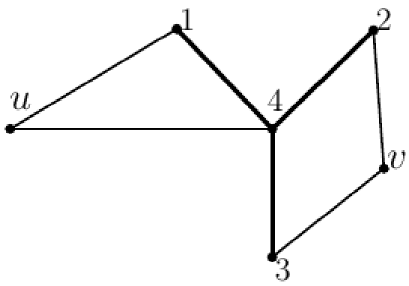


Fig 2.9 : $C_{3(u)}[0, 0, C_4]$

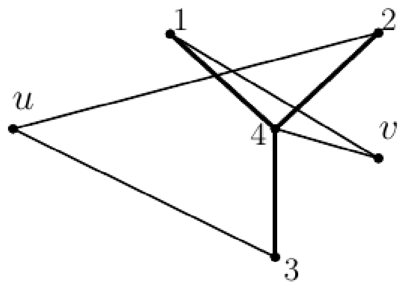


Fig 2.10 : $C^{\sigma}_{3(u)}[0, 0, C_4]$

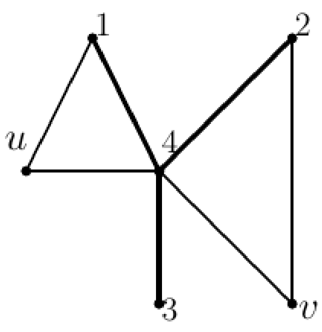


Fig 2.11 : $C_{3(u)}[0, 0, P_2 \cup C_3]$

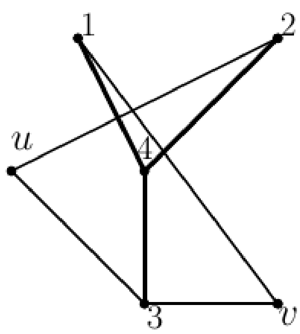


Fig 2.12 : $C^{\sigma}_{3(u)}[0, 0, P_2 \cup C_3]$

The switching graphs of $A(T_4), C_{3(v)}[0, P_2, C_3], C_{3(u)}[0, C_4, 0], C_{3(u)}[0,0, C_4]$ and $C_{3(u)}[0,0, P_2 \cup C_3]$ are given in figures 2.4,2.6,2.8,2.10 and 2.12.

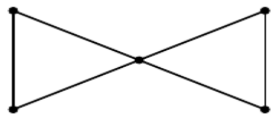
It is evident that the graphs $A(T_4), C_{3(u)}[0, C_4, 0]$ and $C_{3(u)}[0,0, C_4]$ are the graphs that obtained from a 2 -vertex self switching at the vetrex set $\sigma = \{u, v\}$.

Case 2. $G - \sigma$ is connected, unicyclic, $\{d_G(u), d_G(v)\} = \{|V(G)| - 4, |V(G)| - 3\}$ and for $|M - \{v\}| = 2$ and $|N - \{u\}| = 1(|M - \{v\}| = 1$ and $|N - \{u\}| = 2)$ either the elements of $M(N)$ do not lie on the cycle of $G - \sigma$ and the unique path between them intersects the cycle in no more than one vertex or one of the elements of $M(N)$, say α, lies on the cycle and the unique path between them intersects the cycle only at vertex α.

Let $d_G(u) = |V(G)| - 3$ and $d_G(v) = |V(G)| - 4$. Let C represent the only cycle in $G - \sigma$.

If $|V(G)| \geq 6$, then $|V(G) - \sigma| \geq 4$. Also $d_G(u) = |V(R)| - 3 \geq 3$ and $d_G(v) = |V(R)| - 4 \geq 2$. Since $d_G(u) \geq 3$, there exist minimum three vertices, say α, Γ and Δ, in $G - \sigma$ implies that $u\alpha, u\Gamma$ and $u\Delta$ are the edges in G. Since $G - \sigma$ remains connected, the paths $\alpha - \Gamma, \Gamma - \Delta$ and $\alpha - \Delta$ can be found in $G-\sigma$. Now the edges $u\alpha, u\Gamma$ and $u\Delta$ and the paths $\alpha-\Gamma, \Gamma-\Delta$ and $\alpha-\Delta$ form minimum three cycles that are distinct from C. This contradicts G is two-cyclic and hence $|V(G)|=5$

Now $|V(G-\sigma)|=3$. Since $G-\sigma$ is unicyclic, $G-\sigma=C_3$. Also $d_G(u)= |V(G)|-3=2$ and $d_G(v)=|V(G)|-4=1$. Since $d_G(u)=2, u$ is neighbor of exactly a pair of vertices, say α and γ, in $V(G)-\sigma$, thus $u\alpha$ and $u\gamma$ are edges in G. Figure 2.13 presents the unique graph of order 5 with exactly two cycles. As this graph lacks any vertex of degree 1 , no connected 5 -vertex graph with exactly two cycles allows a 2 -vertex self-switching



$Fig\ 2.13$

Case 3. $G-\sigma$ is connected, two-cyclic and $d_G(u)=d_G(v)=|V(G)|-3$

$G-\sigma$ is two-cyclic implies that $|V(G-\sigma)|\geq 5$ and so $|V(G)\geq 7$. Now $d_G(u)=d_G(v)=|V(G)|-3\geq 4$. Since $d_G(u)\geq 4$, there exist minimum four vertices, say α', β', γ' and δ' in $V(G)-\sigma$, and for which the edges $u\alpha'$, $u\beta', u\gamma'$ and $u\delta'$ belong to G. Since $G-\sigma$ contains exactly two cycles, label these cycles as $C1$ and $C2$, which are also in G. Now the edges $\alpha' u$ and $u\beta'$ together with the $\beta'-\alpha'$ path create a cycle in G distinct from $C1$ and $C2$. This contradicts G is two-cyclic and so the case does not exist.

Conversely, Consider a connected graph G such that $|V(G)|=p\geq 5$ with exactly two cycles, where $\sigma=\{u,v\}\subseteq V(G)$ be non-empty for which $G-\sigma$ remains connected and for $uv\notin E(G), G$ is either $A(T_4), C_{3(u)}[0,C_4,0]$ or $C_{3(u)}[0,0,C_4]$ with $d_G(u)=d_G(v)=2$.

It is evident for every graph G, the set $\sigma=\{u,v\}$ constitutes a 2 -vertex self switching of G.

3 Conclusion

This article establishes a condition for a graph G such that G^σ is disconnected and two-cyclic. Furthermore, we provide a characterization of graphs that are connected and two-cyclic that allow a 2 -vertex self-switching.

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