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3-Vertex Duplication Self Switching of Graphs

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Abstract: For a graph $G(V,E)$, duplication of a vertex u of a graph G produces new graph $D(uG)$ by adding a new vertex u' such that $N(u')=N(u)$. A k-vertex duplication of a graph G produces new graph $D((u_1,u_2,\ldots,u_k)G)$ by adding k new vertices $u_1{}',u_2{}',\ldots,u_k{}'$ as the duplication of any k vertices $u_1,u_2,\ldots,u_k$ of G such that $N(u_i')=N(u_i)$. $\sigma=\{u_1,u_2,\ldots,u_k\}\subseteq V(G)$ is called a k-vertex duplication self switching of graph G if $D(\sigma G)\cong D(\sigma G)^{\sigma}$. The set of all k-vertex duplication self switching of G is denoted by $dSS_k(G)$ and the number of elements in the set is denoted by $dss_k(G)$. When k =3, it is called as 3-Vertex Duplication Self Switching. In this paper, we provide the necessary and sufficient conditions needed for a graph to be 3-vertex duplication self switching. We also find $dss_3(G)$ of path, cycle and complete graph.

Keywords: Switching, self vertex switching, 3-vertex self switching, dupli- cation self switching.
Subject Classification Number: *05C60,05C07*

1 Introduction

For a finite undirected Graph $G(V,E)$ with $|V|=p$ and a non-empty set $\sigma\subseteq V$, the switching of G by σ is defined as the graph $G^{\sigma}(V,E')$, which is obtained from G by removing all edges between σ and its complement $V-\sigma$ and adding as edges all non-edges between σ and $V-\sigma$. Switching was defined by Seidel, which is also known as seidel switching [1]. A subset σ of $V(G)$ is said to be self switching if $G\cong G^{\sigma}$ and it is called as $|\sigma|$-vertex self switching. When $|\sigma|$=3, it is called as 3-vertex self switching [5]. The concept of k-vertex duplication self switching was introduced by Jayasekaran and Athithiya [4]. A k-vertex duplication of a graph G produces a new graph $D\big((u_1,u_2,\ldots,u_k)G\big)$ by adding k new vertices $u_1{}',u_2{}',\ldots,u_k{}'$ as the duplication of any k vertices $u_1,u_2,\ldots,u_k$ of G such that $N(u_i')=N(u_i)$. A subset $\sigma=\{u_1,u_2,\ldots,u_k\}$ of $V(G)$ is called a k-vertex duplication self switching of graph G if $D(\sigma G)\cong D(\sigma G)^{\sigma}$. The set of all k-vertex duplication self switching is denoted by $dSS_k(G)$ and the number of elements in the set is denoted by $dss_k(G)$. When k =1, it is called as duplication self vertex switching. The concept of duplication self vertex switching was introduced by Jayasekaran and Prabavathy [3]. When k=3, it is called as 3-vertex duplication self switching. In this paper, we provide the necessary and sufficient conditions needed for a graph to be 3-vertex duplication self switching. We also find $dss_3(G)$ of path, cycle and complete graph.

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2 Preliminaries

Definition 2.1. [4] A k-vertex duplication of a graph G produces a new graph G' by adding k new vertices $u_1{}',u_2{}',\ldots,u_k{}'$ as the duplication of any k vertices $u_1,u_2,\ldots,u_k$ of G such that $N(u_i')=N(u_i)$. The new graph obtained after duplication of the k vertices is denoted by $D((u_1,u_2,\ldots,u_k)G)$. If $\sigma=\{u_1,u_2,\ldots,u_k\}$, then the duplication of G by σ is denoted by $D(\sigma G)$.

Definition 2.2. [4] $\sigma = \{u_1, u_2, \dots., u_k\} \subseteq V(G)$ is called a k-vertex duplication self switching of a graph G if $D(\sigma G) \cong D(\sigma G)^\sigma$. The set of all k-vertex duplication self switching of G is denoted by $dSS_k(G)$. The number of k-vertex duplication self switchings is denoted by $dss_k(G)$.

Definition 2.3. [3] If v is a duplication self vertex switching of a graph G of order p, then p is even and $\deg_G(v) = \frac{p}{2}$.

Definition 2.4. [5] If σ is a 3-vertex self switching of G, then

$$deg_G(u) + deg_G(v) + deg_G(w) = \begin{cases} \frac{3p-5}{2} & if\ G[\sigma] = K_1 \cup K_2 \\ \frac{3p-1}{2} & if\ G[\sigma] = P_3 \\ \frac{3p+3}{2} & if\ G[\sigma] = K_3 \\ \frac{3p-9}{2} & if\ G[\sigma] = \overline{K_3} \end{cases}$$

Definition 2.5. [5] Let G be a graph and let $\sigma = \{u, v, w\} \subset V(G)$ be a 3-vertex self switching of G. Then the number of edges between the vertices of σ and $V - \sigma$ in G is $\frac{3(p-3)}{2}$.

Definition 2.1. [6] If G is a tree with order p and size q, then $q = p - 1$.

3 Main Results

Definition 3.1. A 3-vertex duplication of a graph G produces a new graph G' by adding three vertices u', v', w' as the duplication of any three vertices u, v, w of G such that all the vertices which are adjacent to u, v, w are also adjacent to u', v', w' respectively. It is denoted by $D((u, v, w)G)$.

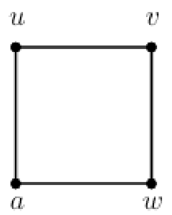

Fig 1. G

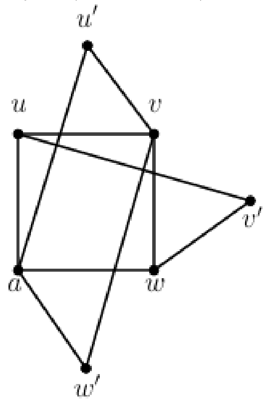

Fig 2. $D((u, v, w)G)$

Definition 3.2. $\sigma = \{u, v, w\} \subseteq V(G)$ is called a 3-vertex duplication self switching of a graph G if $D(\sigma G) \cong D(\sigma G)^\sigma$. The set of all 3-vertex duplication self switching of G is denoted by $dSS_3(G)$. The number of 3-vertex duplication self switchings is denoted by $dss_3(G)$.

Example 3.3. Consider the graph G given in figure 3. Let $\sigma = \{u, v, w\} \subseteq V(G)$. The graphs $D(\sigma G)$ and $D(\sigma G)^\sigma$ are given in figure 4 and figure 5 respectively. Clearly, $D(\sigma G) \cong D(\sigma G)^\sigma$ and hence σ is a 3-vertex duplication self switching of graph G.

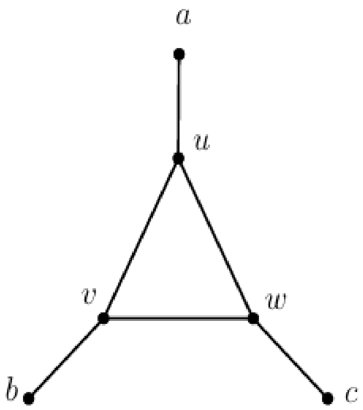

Fig 3. G

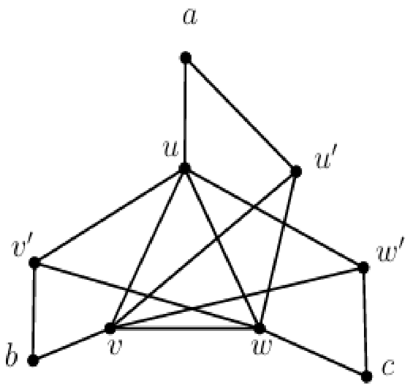

Fig 4. $D(\sigma G)$

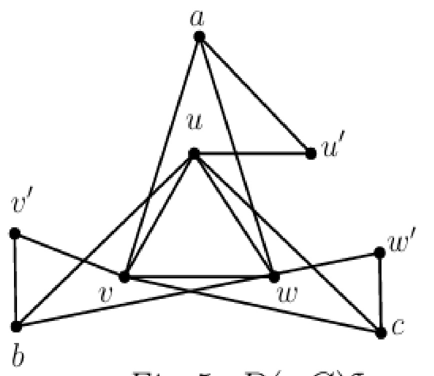

Fig 5. $D(\sigma G)^{\sigma}$

Result 3.4. Let G be a (p,q) graph. Then $D\big((u,v,w)G\big)$ is a $(p+3,\ deg_G(u)+deg_G(v)+deg_G(w))$ graph.

Theorem 3.5. Let $\sigma=\{u,v,w\}\subseteq V(G)$. Then $\deg_{D(\sigma G)}(u)+deg_{D(\sigma G)}(v)+\deg_{D(\sigma G)}(w)\ =\deg_G(u)+deg_G(v)+\deg_G(w)+i$ where $i=0,2,4,6$ for $G[\sigma]=\overline{K_3}, K_1\cup K_2, P_3, K_3$, respectively.

Proof. Let G be the given graph. Let $\sigma=\{u,v,w\}\subseteq V(G)$. Let $D(\sigma G)$ be a 3-vertex duplication of graph G. Let u',v',w' be the duplication vertices of u,v and w respectively. We consider the following four cases.

Case 1. $G[\sigma]=\overline{K_3}$

u is non-adjacent to both v and w in G implies that u' is non-adjacent to both v and w in $D(\sigma G)$ and so $deg_{D(\sigma G)}(u)=deg_G(u)$. Similarly, $deg_{D(\sigma G)}(v)=deg_G(v)$ and $deg_{D(\sigma G)}(w)=deg_G(w)$. Hence, $deg_{D(\sigma G)}(u)+deg_{D(\sigma G)}(v)+deg_{D(\sigma G)}(w)=\ deg_G(u)+deg_G(v)+deg_G(w)$.

Case 2. $G[\sigma]=\ K_1\cup K_2$

Let K_2 be uv and consider w be the vertex of K_1 which is not adjacent to both u and v. Since u is adjacent to v in G, u' is adjacent to v in $D(\sigma G)$. Similarly, v' is adjacent to u in $D(\sigma G)$. Therefore, $deg_{D(\sigma G)}(u)=deg_G(u)+1$ and $deg_{D(\sigma G)}(v)=deg_G(v)+1$. Since w is non-adjacent to both u and v, the vertex w' is non-adjacent to both u and v and so $deg_{D(\sigma G)}(w)=deg_G(w)$. Hence $\deg_{D(\sigma G)}(u)+deg_{D(\sigma G)}(v)+\deg_{D(\sigma G)}(w)=\deg_G(u)+deg_G(v)+\deg_G(w)\ +2$.

Case 3. $G[\sigma]=P_3$

Let P_3 be uvw. Since v is adjacent to both u and w in G, v' is adjacent to both u and w in $D(\sigma G)$. As, u and w are adjacent to v in G, both w' and u' are adjacent to v in $D(\sigma G)$. This implies that $deg_{D(\sigma G)}(u)=deg_G(u)+$

1,$deg_{D(\sigma G)}(w)$=$deg_G(w)+1$, $deg_{D(\sigma G)}(v)$= $deg_G(v)+2$. Hence $\deg_{D(\sigma G)}(u)+deg_{D(\sigma G)}(v)+\deg_{D(\sigma G)}(w)=deg_G(u)+deg_G(v)+deg_G(w)\ +4$.

Case 4. $G[\sigma]=K_3$

Since u is adjacent to both v and w in G, u' is adjacent to both v and w in $D(\sigma G)$. Similarly, v' is adjacent to both u and w, w' is adjacent to both u and v in $D(\sigma G)$. This implies that $deg_{D(\sigma G)}(u)$= $deg_G(u)+2$, $deg_{D(\sigma G)}(v)$= $deg_G(v)+2$, $deg_{D(\sigma G)}(w)$= $deg_G(w)+2$. Hence $\deg_{D(\sigma G)}(u)+deg_{D(\sigma G)}(v)+\deg_{D(\sigma G)}(w)=\deg_G(u)+deg_G(v)+\deg_G(w)\ +6$.

Hence theorem follows from above four cases.

Theorem 3.6. If σ is a 3-vertex duplication self switching of G, then

$deg_G(u)+deg_G(v)+deg_G(w)=\frac{3p}{2}$.

Proof. Let $\sigma=\{u,v,w\}$ be a 3-vertex duplication self switching of G. Then σ is a 3-vertex self switching of the graph $D(\sigma G)$ of order p+3. Now, $D(\sigma G)[\sigma]$ is either $\overline{K_3}$ or $K_1\cup K_2$ or P_3 or K_3.

Case 1. $D(\sigma G)[\sigma]=\overline{K_3}$

By Theorem 2.4, $\deg_{D(\sigma G)}(u)+deg_{D(\sigma G)}(v)+\ deg_{D(\sigma G)}(w)=\frac{3(p+3)-9}{2}=\frac{3p}{2}$. Since $G[\sigma]=D(\sigma G)[\sigma]$, $G[\sigma]=\overline{K_3}$. By Theorem 3.5, $deg_G(u)+deg_G(v)+deg_G(w)=\frac{3p}{2}$.

Case 2. $D(\sigma G)[\sigma]=\ K_1\cup K_2$

Let K_2 be uv and consider w be the vertex of K_1 which is not adjacent to both u and v. By Theorem 2.4, we have, $\deg_{D(\sigma G)}(u)+deg_{D(\sigma G)}(v)+deg_{D(\sigma G)}(w)=\frac{3(p+3)-5}{2}=\frac{3p+4}{2}$. Since $G[\sigma]=D(\sigma G)[\sigma]$, $G[\sigma]=\ K_1\cup K_2$. By Theorem 3.5, $deg_G(u)+deg_G(v)+deg_G(w)=\frac{3p+4}{2}$ which implies $deg_G(u)+deg_G(v)+deg_G(w)=\frac{3p}{2}$.

Case 3. $D(\sigma G)[\sigma]=P_3$

By Theorem 2.4, $\deg_{D(\sigma G)}(u)+deg_{D(\sigma G)}(v)+\ deg_{D(\sigma G)}(w)=\frac{3(p+3)-1}{2}=\frac{3p+8}{2}$. Since $G[\sigma]=D(\sigma G)[\sigma]$, $G[\sigma]=P_3$. By Theorem 3.5, $deg_G(u)+deg_G(v)+deg_G(w)=\frac{3p+8}{2}$ which implies $deg_G(u)+deg_G(v)+deg_G(w)=\frac{3p}{2}$.

Case 4. $D(\sigma G)[\sigma]=\ K_3$

By Theorem 2.4, $deg_{\{D(\sigma G)\}}(u)+deg_{\{D(\sigma G)\}}(v)deg_{\{D(\sigma G)\}}(w)=\frac{3(p+3)+3}{2}=\frac{3p+12}{2}$. Since $G[\sigma]=D(\sigma G)[\sigma]$, $G[\sigma]=K_3$. By Theorem 3.5, $deg_G(u)+deg_G(v)+deg_G(w)=\frac{3p+12}{2}$ which implies $deg_G(u)+deg_G(v)+deg_G(w)=\frac{3p}{2}$.

Hence theorem follows from above four cases.

Remark 3.7. The converse of the above theorem is not necessarily true.

Consider the cycle graph $G=C_4$ given in figure 6. Let $\sigma=\{u,v,w\}\subseteq V(G)$. Here $deg_G(u)+deg_G(v)+deg_G(w)\ =2+2+2=6=\frac{3(4)}{2}=\frac{3p}{2}$. The graph $D(\sigma G)$ and $D(\sigma G)^\sigma$ are given in figure 7 and figure 8 respectively. Clearly, $D(\sigma G)$ is not isomorphic to $D(\sigma G)^\sigma$ and hence σ is not a 3-vertex duplication self switching of G.

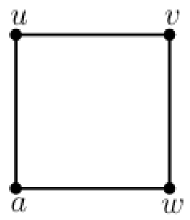

Fig 6. G

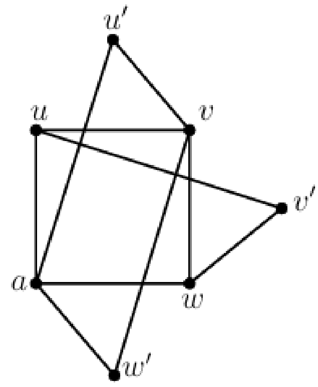

Fig 7. $D(\sigma G)$

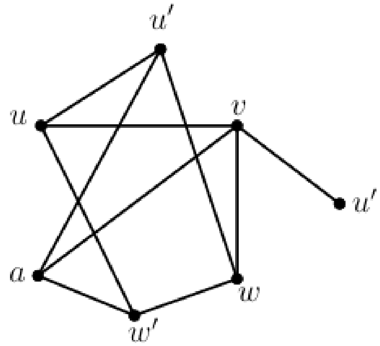

Fig 8. $D(\sigma G)^{\sigma}$

Theorem 3.8. Let G be a (p,q) graph and let $\sigma \subseteq V(G)$ be a 3-vertex duplication self switching of G. Then the number of edges between the vertices of σ and $V\big(D(\sigma G)\big) - \sigma$ in $D(\sigma G)$ is $\frac{3p}{2}$.

Proof. Let $\sigma \subseteq V(G)$ be a 3-vertex duplication self switching of G. Then σ is a 3-vertex self switching of $D(\sigma G)$ of order $p+3$ and so $D(\sigma G) \cong D(\sigma G)^{\sigma}$. By Theorem 2.5, the number of edges between the vertices of σ and $V\big(D(\sigma G)\big) - \sigma$ in $D(\sigma G)$ is $\frac{3(p+3-3)}{2} = \frac{3p}{2}$.

Corollary 3.9. If a graph has a 3-vertex duplication self switching, then the order of the graph is even.

Proof. Let G be a (p,q) graph and $\sigma \subseteq V(G)$ be a 3-vertex duplication self switching of G. By Theorem 3.8, $\frac{3p}{2}$ is an integer and therefore p is even.

Theorem 3.10. If G is a graph with odd size, then the line graph $L(G)$ has no 3-vertex duplication self switching.

Proof. Let G be a graph with odd size. By definition of line graph $L(G)$, edges of G are the vertices of $L(G)$. Since G has odd number of edges, $L(G)$ has odd number of vertices. By Corollary 3.9, $L(G)$ has no 3-vertex duplication self switching.

Theorem 3.11. Let G be a self complementary graph with $|V(G)| = 4n, n \in N$ and n is odd. Then $L(G)$ and $L(\bar{G})$ have no 3-vertex duplication self switching.

Proof. Let G be a self complementary graph with $|V(G)| = 4n, n \in N$ and n is odd. Then $G \cong \bar{G}$ and $|E(G)| = |E(\bar{G})|$. Since $|E(G)| + |E(\bar{G})| = \binom{p}{2}$, $2|E(G)| = \frac{4n(4n-1)}{2}$ and so $|E(G)| = n(4n-1)$. Hence both G and $\bar{G}$ have odd number of edges and so both L(G) and L(G) have odd number of vertices. By Corollary 3.9, $L(G)$ and $L(\bar{G})$ have no 3-vertex duplication self switching.

Theorem 3.12. Let G be (p,q) tree with p is even. Then $L(G)$ has no 3-vertex duplication self switching.

Proof. Let G be a tree with even order p. By

Theorem 3.6, $q = p - 1$ is odd. By Theorem 3.10, $L(G)$ has no 3-vertex duplication self switching.

Theorem 3.13. If eulerian graph has a 3-vertex duplication self switching, then the order of the graph is multiple of 4.

Proof. Let G be an eulerian graph and $\sigma = \{u, v, w\} \subseteq V(G)$ be a 3-vertex duplication self switching of G. By Theorem 3.6, $deg_G(u) + deg_G(v) + deg_G(w) = \frac{3p}{2}$. Since G is eulerian, all vertices are of even degree. This implies that $\frac{3p}{2}$ is even and therefore $p = 4n, n \in N$.

Theorem 3.14. $dss_3(P_p) = 0$ for $p \geq 3$

Proof. Let $G = P_p$ be the path graph with p vertices and $p - 1$ edges. Let $\sigma = \{u, v, w\} \subseteq V(G)$ be a 3-vertex duplication self switching of G. By Theorem 3.6, $deg_G(u) + deg_G(v) + deg_G(w) = \frac{3p}{2}$. If $p \geq 6$, then $deg_G(u) + deg_G(v) + deg_G(w) = \frac{3p}{2} \geq \frac{3(6)}{2} = 9$. But for any three vertices u, v and w in P_p, $deg_G(u) + deg_G(v) + deg_G(w) \leq 6$ which is a contradiction. Hence, $dss_3(G) = 0$ for $p \geq 6$. So we calculate $dss_3(G)$ for $3 \leq p \leq 5$. If $p \in \{3,5\}$, then by Corollary 3.9, G has no 3-vertex duplication self switching. Let us calculate $dss_3(G)$ for $p = 4$ But there is no possibility of choosing three vertices u, v, w such that $deg_G(u) + deg_G(v) + deg_G(w) = 6 = \frac{3p}{2}$. Hence, $dss_3(P_p) = 0$.

Theorem 3.15. $dss_3(C_p) = 0$ for $p \geq 3$

Proof. Let $G = C_p$ be the cycle graph with p vertices and p edges. Let $\sigma = \{u, v, w\} \subseteq V(G)$. Since C_p is a 2-regular graph, for any three vertices u, v and w in C_p, $deg_G(u) + deg_G(v) + deg_G(w) = 6$. If p is odd, then by Corollary 3.9, G has no 3-vertex duplication self switching. Let us calculate $dss_3(G)$ for even number of vertices. Now, $deg_G(u) + deg_G(v) + deg_G(w) = 6 = \frac{3p}{2}$ for $p = 4$ only.

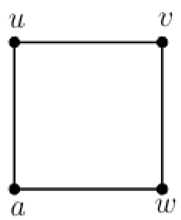

Fig 9. $G = C_4$

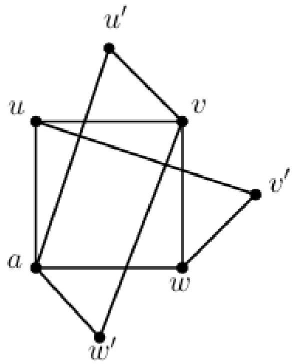

Fig 10. $D(\sigma G)$

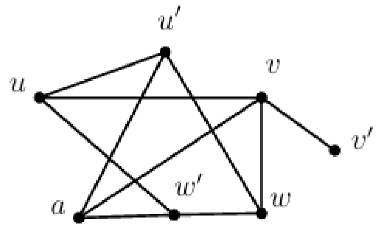

Fig 11. $D(\sigma G)^\sigma$

The graphs $D(\sigma G)$ and $D(\sigma G)^\sigma$ are given in figure 10 and figure 11 respectively. In $D(\sigma G)^\sigma$, the vertex v' has degree one whereas $D(\sigma G)$ has no vertex of degree one. Clearly, $D(\sigma G)$ is not isomorphic to $D(\sigma G)^\sigma$ and so σ is not a 3-vertex duplication self switching of G. Hence, $dss_3(C_p) = 0$.

Theorem 3.16. $dss_3(K_p) = 0$ for $p \geq 3$

Proof. Let $G = K_p$ be the complete graph with p vertices. Let $\sigma = \{u, v, w\} \subseteq V(G)$ so that $p \geq 3$. Since degree of each vertex in K_p is $p - 1$, for any three vertices u, v and w in K_p, $deg_G(u) + deg_G(u) + deg_G(w) = 3p - 3$. If p is odd, then by Corollary 3.9, G has no 3-vertex duplication self switching. Let us calculate $dss_3(G)$ for even number of vertices. If σ is a 3-vertex duplication self switching of G, then $deg_G(u) + \mathrm{deg}_G(v) + deg_G(w) = \frac{3p}{2}$ and so $3p - 3 = \frac{3p}{2}$ implies that $p = 2$ which is contradiction. Hence, $dss_3(K_p) = 0$.

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