International Journal of
INTELLIGENT SYSTEMS AND APPLICATIONS IN ENGINEERING
ISSN:2147-6799 www.ijisae.org Original Research Paper

Twain Secure Perfect Dominating Sets of Tadpole $(T_{n,1})$ Graphs

[1]K. Lal Gipson and [2]C. Vinisha

Submitted: 17/05/2024 **Revised:** 20/06/2024 **Accepted:** 12/07/2024

Abstract: Let $G = (V, E)$ be a simple graph. A set $S \subseteq V$ is a dominating set of G, if every vertex in $V \setminus S$ is adjacent to a vertex in S. A subset S of V is called a twain secure perfect dominating set of G (TSPD- set) if every vertex $v \in V \setminus S$ is adjacent to exactly one vertex $u \in S$ and $(S \setminus \{u\}) \cup \{v\}$ is a dominating set of G. The minimum cardinality of a twain secure perfect dominating set of G is called the twain secure perfect domination number of G and is denoted by $\gamma_{tsp}(G)$. Let $D_{tsp}(T_{n,1}, i)$ be the family of all twain secure perfect dominating sets of $T_{n,1}$ with cardinality i, for $\gamma_{tsp}(T_{n,1}) \leq i \leq n$. In this paper, we construct all the twain secure perfect dominating sets of tadpole graphs $(T_{n,1})$ by recursive method.

Keywords: *tadpole* $(T_{n,1})$, *twain secure perfect dominating set, twain secure perfect domination number.*

Mathematics Subject Classification: 05C69, 05C31

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic definitions and terminologies we refer to [2]. Two vertices u and v are said to be *adjacent* if uv is an edge of G. The *open neighborhood* of a vertex v in a graph G is defined as the set $N_G(V) = \{u \in V(G) : uv \in E(G)\}$, while the *closed neighborhood* of v in G is defined as $N_G[V] = N_G(V) \cup \{v\}$. A subset $S \subseteq V(G)$ is called a *dominating set* if every vertex $v \in V(G) \setminus S$ is adjacent to a vertex $u \in S$. The domination number, $\gamma(G)$, of a graph G denotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph G is hence often called as a γ - set of G [1]. A dominating set S is called a *secure dominating set* if for each $v \in V \setminus S$ there exists $u \in N(v) \cap S$ such that $(S \setminus \{u\}) \cup \{v\}$ is a dominating set. The secure domination number $\gamma_s(G)$ is the minimum cardinality of a secure dominating set of G. The concept of secure domination was introduced by Cockayne et al [3]. A dominating set S is called a *perfect dominating set* if every vertex in $V \setminus S$ is adjacent to exactly one vertex in S. The perfect domination number $\gamma_s(G)$ is the minimum cardinality of a perfect dominating set of G. The concept of perfect domination was introduced by Weichsel [10]. In this sequel, we introduced the concept of twain secure perfect domination of graphs. A dominating set S is called a *twain secure perfect dominating set* of G (TSPD-set) if for every vertex $v \in V \setminus S$ is adjacent to exactly one vertex $u \in S$ and $(S \setminus \{u\}) \cup \{v\}$ is a dominating set of G. The minimum cardinality of a twain secure perfect dominating set of G is called the twain secure perfect domination number of G and is denoted by $\gamma_{tsp}(G)$. Let $D_{tsp}(G, i)$ be the family of all twain secure perfect dominating sets of G with cardinality i. A tadpole $T_{m,n}$ is a graph obtained by appending a path P_n to a cycle C_m with a bridge. In particular $T_{n,1}$ is the tadpole graph with $n + 1$ vertices obtained by joining a path P_1 to a cycle C_n using a bridge.

The families of the twain secure perfect dominating sets of $T_{n,1}$ are built using a recursive techniques in the following section.

For the smallest integer lower than or equal to x, we use $\lfloor x \rfloor$ as normal. We refer to the set $\{1, 2, ..., n\}$ in this article as $[n]$.

2. Main Results

The family of twain secure perfect dominating sets of $T_{n,1}$ with cardinality i is denoted by $D_{tsp}(T_{n,1}, i)$. Also, twain secure perfect dominating sets of $T_{n,1}$ will be examined. The following lemmas are

[1]*Assistant professor, Department of Mathematics & Research Centre, Scott Christian College (Autonomous), Nagercoil - 629 003, Kanyakumari District, Tamil Nadu, India.*
[2]*Research Scholar, Reg. No: 22113162092007, Department of Mathematics & Research Centre, Scott Christian College (Autonomous), Nagercoil - 629 003, Kanyakumari District, Tamil Nadu, India.*
(Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627 012, Tamil Nadu, India.)
Email: [1]lalgipson@yahoo.com, [2]cvinisha1999@gmail.com

necessary to support the primary findings of this article.

Lemma 2.1. For every $n \geq 4$, $\gamma_{tsp}(T_{n,1}) = \begin{cases} \frac{n+3}{2} & for \quad n \equiv 3(mod\ 4) \\ \left\lfloor \frac{n+2}{2} \right\rfloor & for \quad n \not\equiv 3(mod\ 4) \end{cases}$

Proof. let $V(T_{n,1}) = \{v_1, v_2, \dots, v_{n-1}, v_n, v_{n+1}\}$. Now we consider four cases.

- $n \equiv 0(mod\ 4)$. Thus $n = 4k \geq 4$. Then $\{v_1, v_4, v_5, v_8, \dots, v_{4k-3}, v_{4k}, v_{4k+1}\}$ is a twain secure perfect dominating set of $2k + 1 = \frac{n+2}{2}$ vertices.
- $n \equiv 1(mod\ 4)$. Thus $n = 4k + 1 \geq 5$. Then $\{v_2, v_3, v_6, v_7, \dots, v_{4k-1}, v_{4k+2}\}$ is a twain secure perfect dominating set of $2k + 1 = \lfloor \frac{n+2}{2} \rfloor$ vertices.
- $n \equiv 2(mod\ 4)$. Thus $n = 4k + 2 \geq 6$. Then $\{v_1, v_2, v_5, v_6, \dots, v_{4k-2}, v_{4k+1}, v_{4k+2}\}$ is a twain secure perfect dominating set of $2k + 2 = \frac{n+2}{2}$ vertices. Thus, the above three cases indicates if $n \not\equiv 3(mod\ 4)$, $\gamma_{tsp}(T_{n,1}) = \lfloor \frac{n+2}{2} \rfloor$.
- $n \equiv 3(mod\ 4)$. In this case we show that $\gamma_{tsp}(T_{n,1}) = \frac{n+3}{2}$. Let $n = 4k + 3$, for some positive integer k. Since $\{v_1, v_2, v_3, v_6, \dots, v_{4k-1}, v_{4k+2}, v_{4k+3}\}$ is a twain secure perfect dominating set with $2k + 3 = \frac{n+3}{2}$ vertices. It follows that $\gamma_{tsp}(T_{n,1}) = \frac{n+3}{2}$.

Therefore from all the cases, for every $n \geq 4$, $\gamma_{tsp}(T_{n,1}) = \begin{cases} \frac{n+3}{2} & for \quad n \equiv 3(mod\ 4) \\ \left\lfloor \frac{n+2}{2} \right\rfloor & for \quad n \not\equiv 3(mod\ 4) \end{cases}$

Lemma 2.2. For every $n \geq 4$,

i. If $n \equiv 3(mod\ 4)$, then $D_{tsp}(T_{n,1}, i) \neq \emptyset$ if and only if $\frac{n+3}{2} \leq i \leq n + 1$.

ii. If $n \not\equiv 3(mod\ 4)$, then $D_{tsp}(T_{n,1}, i) \neq \emptyset$ if and only if $\left\lfloor \frac{n+2}{2} \right\rfloor \leq i \leq n + 1$.

Proof.

i. Assume that $n \equiv 3(mod\ 4)$. By Lemma 2.1, $\gamma_{tsp}(T_{n,1}) = \frac{n+3}{2}$. Obviously, the maximum cardinality of the twain secure perfect dominating set of tadpole graph with $n + 1$ vertices is $n + 1$. Therefore, $D_{tsp}(T_{n,1}, i) \neq \emptyset$ if and only if $\frac{n+3}{2} \leq i \leq n + 1$.

iii. Assume that $n \not\equiv 3(mod\ 4)$. By Lemma 2.1, $\gamma_{tsp}(T_{n,1}) = \lfloor \frac{n+2}{2} \rfloor$. Obviously the maximum cardinality of the twain secure perfect dominating set of tadpole graph with $n + 1$ vertices is $n + 1$. Therefore, $D_{tsp}(T_{n,1}, i) \neq \emptyset$ if and only if $\left\lfloor \frac{n+2}{2} \right\rfloor \leq i \leq n + 1$.

Lemma 2.3. If $D_{tsp}(T_{n,1}, i) \neq \emptyset$, then

i. $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$, $D_{tsp}(T_{n-2,1}, i-1) = \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) = \emptyset$ if and only if $i = n + 1$.

ii. $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$, $D_{tsp}(T_{n-2,1}, i-1) \neq \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) = \emptyset$ if and only if $i = n$.

iii. $D_{tsp}(T_{n-1,1}, i-1) = \emptyset$, $D_{tsp}(T_{n-2,1}, i-1) \neq \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) \neq \emptyset$ if and only if $n = 4k, i = 2k + 1, k \in N$.

iv. $D_{tsp}(T_{n-1,1}, i-1) = \emptyset$, $D_{tsp}(T_{n-2,1}, i-1) = \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) \neq \emptyset$ if and only if $n = 4k + 1, i = 2k + 1, k \in N$.

Proof.

i. First let us assume that $n \equiv 3(mod\ 4)$.
Since $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$, by Lemma 2.2, $\frac{n+2}{2} \leq i - 1 \leq n$. Which implies

$$i \leq n + 1 \quad (1)$$

Since $D_{tsp}(T_{n-2,1}, i-1) = \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) = \emptyset$, $i - 1 > n - 1$ or

$i - 1 < \frac{n+1}{2}$ and $i - 2 > n - 3$ or $i - 2 < \frac{n-1}{2}$. If $-2 < \frac{n-1}{2}$, then

$i < \frac{n-1}{2} + 2 = \frac{n+3}{2}$. Therefore $i < \frac{n+3}{2}$ holds. Therefore, by Lemma 2.2, $D_{tsp}(T_{n,1}, i) = \emptyset$. Which is a contradiction. So, we have $i - 1 > n - 1$. Which

gives

$i \geq n + 1$ (2)

From (1) and (2), $i = n + 1$.

Conversely, assume that $i = n + 1$. That implies $i - 1 > n - 1$. By Lemma 2.2, $D_{tsp}(T_{n-2,1}, i-1) = \emptyset$. Similarly, $D_{tsp}(T_{n-4,1}, i-2) = \emptyset$. Since $i = n + 1$, by Lemma 2.2, $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$.

Now, assume that $n \not\equiv 3(mod\ 4)$.

Since $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$, by Lemma 2.2, $\lfloor\frac{n+1}{2}\rfloor \leq i - 1 \leq n$. That gives

$i \leq n + 1$ (3)

Since $D_{tsp}(T_{n-2,1}, i-1) = \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) = \emptyset$, $i - 1 > n - 1$ or $i - 1 < \lfloor\frac{n}{2}\rfloor$ and $i - 2 > n - 3$ or $i - 2 < \lfloor\frac{n-2}{2}\rfloor$. Therefore $i - 1 > n - 1$ or $i - 2 < \lfloor\frac{n-2}{2}\rfloor$. Therefore, $i < \lfloor\frac{n+2}{2}\rfloor$ holds. Therefore, by Lemma 2.2,

$D_{tsp}(T_{n,1}, i) = \emptyset$. Which is a contradiction. So, we have $i - 1 > n - 1$. Which gives

$i \geq n + 1$ (4)

From (3) and (4), $i = n + 1$.

Conversely, assume that $i = n + 1$. That implies $i - 1 > n - 1$. By Lemma 2.2, $D_{tsp}(T_{n-2,1}, i-1) = \emptyset$. Similarly, $D_{tsp}(T_{n-4,1}, i-2) = \emptyset$. Since $i = n + 1$, by Lemma 2.2, $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$.

ii. First let us assume that $n \equiv 3(mod\ 4)$.

Since $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$, $D_{tsp}(T_{n-2,1}, i-1) \neq \emptyset$, $\frac{n+2}{2} \leq i - 1 \leq n$ and $\frac{n+1}{2} \leq i - 1 \leq n - 1$. Which implies $\frac{n+2}{2} \leq i - 1 \leq n - 1$. Which gives

$i \leq n$. (5)

Since $D_{tsp}(T_{n-4,1}, i-2) = \emptyset$, by Lemma 2.2, $i - 2 > n - 3$ or $i - 2 < \frac{n-1}{2}$.

If $i - 2 < \frac{n-1}{2}$, then $i < \frac{n+3}{2}$. Therefore, $i < \frac{n+3}{2}$ holds. By Lemma 2.2,

$D_{tsp}(T_{n,1}, i) = \emptyset$. Which is a contradiction. Therefore $i - 2 > n - 3$. Which gives

$i \geq n$ (6)

From (5) and (6), $i = n$.

Conversely, assume that $i = n$. Which implies $i - 2 > n - 3$. By Lemma 2.2, $D_{tsp}(T_{n-4,1}, i-2) = \emptyset$. Since $i = n, i - 1 = n - 1$. By Lemma 2.2,

$D_{tsp}(T_{n-2,1}, i-1) \neq \emptyset$. Similarly, $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$.

Next assume that $n \not\equiv 3(mod\ 4)$.

Since $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$, $D_{tsp}(T_{n-2,1}, i-1) \neq \emptyset$, by Lemma 2.2, $\left\lfloor\frac{n+1}{2}\right\rfloor \leq i - 1 \leq n$ and $\lfloor\frac{n}{2}\rfloor \leq i - 1 \leq n - 1$. Therefore, $\left\lfloor\frac{n+1}{2}\right\rfloor \leq i - 1 \leq n - 1$.

That gives

$i \leq n$ (7)

Since $D_{tsp}(T_{n-4,1}, i-2) = \emptyset$, $i - 2 > n - 3$ or $i - 2 < \lfloor\frac{n-2}{2}\rfloor$. If $i - 2 < \lfloor\frac{n-2}{2}\rfloor$, then $i < \lfloor\frac{n+2}{2}\rfloor$ holds. Therefore, by Lemma 2.2, $D_{tsp}(T_{n,1}, i) = \emptyset$. Which is a contradiction. Therefore $i - 2 > n - 3$. Which gives

$i \geq n$ (8)

From (7) and (8), $i = n$.

Conversely, assume that $i = n$. Which implies $i - 2 > n - 3$. By Lemma 2.2, $D_{tsp}(T_{n-4,1}, i-2) = \emptyset$. Since $i = n, i - 1 = n - 1$. By Lemma 2.2,

$D_{tsp}(T_{n-2,1}, i-1) \neq \emptyset$. Similarly, $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$.

iii. Let us consider $n \equiv 0 (mod\ 4)$. Since $D_{tsp}(T_{n-1,1}, i-1) = \emptyset$, $i-1 > n$ or $i-1 < \lfloor\frac{n+1}{2}\rfloor$. If $i-1 > n$, then $i > n+1$. By Lemma 2.2, $D_{tsp}(T_{n,1}, i) = \emptyset$. Which is a contradiction. So $i-1 < \lfloor\frac{n+1}{2}\rfloor$. Which implies

$$i < \left\lfloor\frac{n+1}{2}\right\rfloor + 1 \quad (9)$$

Since $D_{tsp}(T_{n-2,1}, i-1) \neq \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) \neq \emptyset$, $\left\lfloor\frac{n}{2}\right\rfloor \leq i-1 \leq n-1$ and $\lfloor\frac{n-2}{2}\rfloor \leq i-2 \leq n-3$. That gives $\lfloor\frac{n}{2}\rfloor \leq i-1 \leq n-2$.

Which implies,

$$\lfloor\frac{n}{2}\rfloor + 1 \leq i \leq n-1 \quad (10)$$

From (9) and (10), $\lfloor\frac{n}{2}\rfloor + 1 \leq i < \left\lfloor\frac{n+1}{2}\right\rfloor + 1$. Which gives $n = 4k, i = 2k+1$, for some $k \in N$.

Conversely, assume that $n = 4k, i = 2k+1$, for some $k \in N$. Now,

$\gamma_{tsp}(T_{n-1}, 1) = \lfloor\frac{n+1}{2}\rfloor = \lfloor\frac{4k+1}{2}\rfloor = \lfloor 2k + \frac{1}{2}\rfloor = \lfloor i - \frac{1}{2}\rfloor > i > i-1$.

Therefore, $i-1 < \left\lfloor\frac{n+1}{2}\right\rfloor$. By Lemma 2.2, $D_{tsp}(T_{n-1,1}, i-1) = \emptyset$.

Now, $\gamma_{tsp}(T_{n-4}, 1) = \left\lfloor\frac{n-2}{2}\right\rfloor = \left\lfloor\frac{4k-2}{2}\right\rfloor = \lfloor 2k-1\rfloor = \lfloor 2k+1-2\rfloor = \lfloor i-2\rfloor = i-2$. Therefore, $i-2 = \lfloor\frac{n-2}{2}\rfloor$. By Lemma 2.2, $D_{tsp}(T_{n-4,1}, i-2) \neq \emptyset$. Similarly, $D_{tsp}(T_{n-2,1}, i-1) \neq \emptyset$.

iv. Let us consider $n \equiv 1 (mod\ 4)$.

Since $D_{tsp}(T_{n-1,1}, i-1) = \emptyset$ and $D_{tsp}(T_{n-2,1}, i-1) = \emptyset$, $i-1 < \lfloor\frac{n+1}{2}\rfloor$ or $i-1 > n$ and $i-1 < \lfloor\frac{n}{2}\rfloor$ or $i-1 > n-1$. Therefore $i-1 > n$ or $i-1 < \lfloor\frac{n}{2}\rfloor$. If $i-1 > n$, then $i > n+1$. By Lemma 2.2, $D_{tsp}(T_{n,1}, i) = \emptyset$. Which is a contradiction. So,

$$i-1 < \lfloor\frac{n}{2}\rfloor \quad (11)$$

Since $D_{tsp}(T_{n-4,1}, i-2) \neq \emptyset$, by Lemma 2.2, $\lfloor\frac{n-2}{2}\rfloor \leq i-2 \leq n-3$. Which gives

$$\left\lfloor\frac{n-2}{2}\right\rfloor + 1 \leq i-1 \leq n-2 \quad (12)$$

From (11) and (12), $\left\lfloor\frac{n-2}{2}\right\rfloor + 1 \leq i-1 < \lfloor\frac{n}{2}\rfloor$. That implies $\left\lfloor\frac{n-2}{2}\right\rfloor + 2 \leq i < \lfloor\frac{n}{2}\rfloor + 1$. Which gives $n = 4k+1, i = 2k+1$, for some $k \in N$.

Conversely, assume that $n = 4k+1, i = 2k+1$, for some $k \in N$.

Now, $\gamma_{tsp}(T_{n-1,1}) = \lfloor\frac{n+1}{2}\rfloor = \lfloor\frac{4k+2}{2}\rfloor = \lfloor 2k+1\rfloor = i > i-1$. Therefore,

$i-1 < \lfloor\frac{n+1}{2}\rfloor$. By Lemma 2.2, $D_{tsp}(T_{n-1,1}, i-1) = \emptyset$. Similarly, $D_{tsp}(T_{n-2,1}, i-1) = \emptyset$.

Now, $\gamma_{tsp}(T_{n-4,1}) = \lfloor\frac{n-2}{2}\rfloor = \lfloor\frac{4k+1-2}{2}\rfloor = \lfloor\frac{4k-1}{2}\rfloor = \lfloor 2k - \frac{1}{2}\rfloor = \lfloor 2k+1 - \frac{3}{2}\rfloor = \lfloor i - \frac{3}{2}\rfloor \leq i-2$. By Lemma 2.2, $D_{tsp}(T_{n-4,1}, i-2) \neq \emptyset$.

Theorem 2.4. For every $n \geq 8$,

i. If $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$, $D_{tsp}(T_{n-2,1}, i-1) = \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) = \emptyset$ then $D_{tsp}(T_{n,1}, i) = \{[n+1]\}$.

ii. If $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$, $D_{tsp}(T_{n-2,1}, i-1) \neq \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) = \emptyset$, then $D_{tsp}(T_{n,1}, i) = \{[n]\}$.

iii. If $D_{tsp}(T_{n-1,1}, i-1) = \emptyset$, $D_{tsp}(T_{n-2,1}, i-1) \neq \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) \neq \emptyset$, then $D_{tsp}(T_{n,1}, i) = \{\{1,4,5,8,9,\ldots,n-4,n-3,n,n+1\},$

$\{3, 4, 7, 8, 9, \dots, n-4, n-1, n, n+1\}\}$.

iv. If $D_{tsp}(T_{n-1,1}, i-1) = \emptyset$, $D_{tsp}(T_{n-2,1}, i-1) = \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) \neq \emptyset$, then $D_{tsp}(T_{n,1}, i) = \{\{1, 4, 5, 8, 9, \dots,, n-3, n-1, n\}, \{2, 3, 6, 7, 10, \dots, n-3, n-2, n+1\}\}$.

v. If $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) \neq \emptyset$ then $D_{tsp}(T_{n,1}, i) = \{\{X \cup \{n+1\}$ if X ends with $n\} \cup \{X \cup \{n\}$ if X ends with $n-1\} \cup \{X \backslash \{n\} \cup \{n-2, n+1\}$ if X ends with n and X starts with $2\} \cup \{Y \cup \{n, n+1\}$ if Y starts with 1 and ends with $n-3\} \cup \{Y \cup \{n-1, n\}$ if Y ends with $n-4\} \cup \{Y \cup \{n-2, n+1\}$ if Y starts with 2 and ends with $n-3\} \cup \{Y \backslash \{n-3\} \cup \{n-1, n, n+1\}$ if Y starts with 3 ends with $n-3\}\}$, where $X \in D_{tsp}(T_{n-1,1}, i-1), Y \in D_{tsp}(T_{n-4,1}, i-2)$.

Proof.

i. Since $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$, $D_{tsp}(T_{n-2,1}, i-1) = \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) = \emptyset$, by Lemma 2.3$(i)$, $i = n+1$. We know that for any $G = (V, E)$, $V(G)$ is always a twain secure perfect dominating set. Hence $D_{tsp}(T_{n,1}, n+1) = \{[n+1]\}$. Thus $D_{tsp}(T_{n,1}, i) = \{[n+1]\}$.

ii. Since $D_{tsp}(T_{n-1,1}, i-1) \neq \emptyset$, $D_{tsp}(T_{n-2,1}, i-1) \neq \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) = \emptyset$, by Lemma 2.3$(ii)$, $i = n$. By the definition of twain secure perfect dominating set if we choose n vertices from $n+1$ vertices the only possible set is $\{1, 2, 3, \dots, n\}$ simply say $[n]$. Hence $D_{tsp}(T_{n,1}, n) = \{[n]\}$. Thus $D_{tsp}(T_{n,1}, i) = \{[n]\}$.

iii. Since $D_{tsp}(T_{n-1,1}, i-1) = \emptyset$, $D_{tsp}(T_{n-2,1}, i-1) \neq \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) \neq \emptyset$, by Lemma 2.3$(iii)$, $n = 4k, i = 2k+1, k \in N$. Clearly the sets $\{1, 4, 5, 8, 9, \dots, n-4, n-3, n, n+1\}$ and $\{3, 4, 7, 8, 9, \dots, n-4, n-1, n, n+1\}$ has $\frac{n+2}{2}$ elements. By the definition of twain secure perfect domination of tadpole graph $1, 4, 5$ cover all the vertices up to 5 and $3, 4, 5$ cover all the vertices up to 5 for $n+1 = 5$. Proceeding like this we obtain that $\{1, 4, 5, 8, 9, \dots, n-4, n-3, n, n+1\}, \{3, 4, 7, 8, 9, \dots, n-4, n-1, n, n+1\}$ cover all the vertices up to $n+1$. The other sets with cardinality $\frac{n+2}{2} = 3$ are $\{1, 2, 5\}, \{1, 3, 4\}$, etc. In the set $\{1, 2, 5\}$, the vertex 4 is adjacent to 2 and 5. In the set $\{1, 3, 4\}$ the vertex 2 is adjacent to 1 and 3. These sets are not satisfied the twain secure perfect domination. So, the only sets $\{1, 4, 5, 8, 9, \dots, n-4, n-3, n, n+1\}$ and $\{3, 4, 7, 8, 9, \dots, n-4, n-1, n, n+1\}$ are twain secure perfect dominating sets.

iv. Since $D_{tsp}(T_{n-1,1}, i-1) = \emptyset$, $D_{tsp}(T_{n-2,1}, i-1) = \emptyset$ and $D_{tsp}(T_{n-4,1}, i-2) \neq \emptyset$, by Lemma 2.4$(iv)$, $n = 4k+1, i = 2k+1$. Clearly the sets $\{1, 4, 5, 8, 9, \dots, n-3, n-1, n\}$ and $\{2, 3, 6, 7, 10, \dots, n-3, n-2, n+1\}$ has $\frac{n+1}{2}$ elements. By the definition of twain secure perfect domination of tadpole graph $1, 4, 5$ and $2, 3, 6$ cover all the vertices up to 6. Proceeding like this we obtain that $\{1, 4, 5, 8, 9, \dots, n-3, n-1, n\}$ and $\{2, 3, 6, 7, 10, \dots, n-3, n-2, n+1\}$ cover all the vertices up to $n+1$. The other sets with cardinality $\frac{n+1}{2} = 5$ are $\{1, 2, 3, 8, 9\}$, etc. In the set $\{1, 2, 3, 8, 9\}$ is not a dominating set. The only sets $\{1, 4, 5, 8, 9, \dots, n-3, n-1, n\}$ and $\{2, 3, 6, 7, 10, \dots, n-3, n-2, n+1\}$ are twain secure perfect dominating sets.

v. Construction of $D_{tsp}(T_{n,1}, i)$ follows from $D_{tsp}(T_{n-1,1}, i-1)$ and $D_{tsp}(T_{n-4,1}, i-2)$. Let X be a twain secure perfect dominating set of $T_{n-1,1}$ with cardinality $i-1$. The elements of $D_{tsp}(T_{n-1,1}, i-1)$ ends with n or $n-1$.

- ♦ If $n-1 \in X$ and $n \notin X$, then the elements of $D_{tsp}(T_{n-1,1}, i-1)$ belongs to $D_{tsp}(T_{n,1}, i)$ by adjoining n.

- If $n \in X$ and the set X starts with the vertex 2, then we removed the vertex n from $D_{tsp}(T_{n-1,1}, i-1)$ and adjoin the vertices $n-2$ and $n+1$ in $D_{tsp}(T_{n-1,1}, i-1)$. Now the elements of $D_{tsp}(T_{n-1,1}, i-1)$ belongs to $D_{tsp}(T_{n,1}, i)$.
- If $n \in X$ and the set X does not start with the vertex 2, then the elements of $D_{tsp}(T_{n-1,1}, i-1)$ belongs to $D_{tsp}(T_{n,1}, i)$ by adjoining $n+1$.

Let Y be a twain secure perfect dominating set of $T_{n-4,1}$ with cardinality $i-2$.

The elements of $D_{tsp}(T_{n-4,1}, i-2)$ ends with $n-4$ or $n-3$. If $n-4 \in Y$ and $-3 \notin Y$, then the elements of $D_{tsp}(T_{n-4,1}, i-2)$ belongs to $D_{tsp}(T_{n,1}, i)$ by adjoining $n-1$ and n.

If $n-3 \in Y$, then we consider three cases.

- Suppose $1 \notin Y$, $2 \notin Y$ and the set Y starts with the vertex 3, then we removed the vertex $n-3$ from $D_{tsp}(T_{n-4,1}, i-2)$ and adjoin the vertices $n-1, n$ and $n+1$ in $D_{tsp}(T_{n-4,1}, i-2)$. Now the elements of $D_{tsp}(T_{n-4,1}, i-2)$ belongs to $D_{tsp}(T_{n,1}, i)$.
- Suppose $1 \notin Y$ and the set Y starts with the vertex 2, then the elements of $D_{tsp}(T_{n-4,1}, i-2)$ belongs to $D_{tsp}(T_{n,1}, i)$ by adjoining $n-2$ and $n+1$.
- Suppose the set Y starts with the vertex 1, then the elements of $D_{tsp}(T_{n-4,1}, i-2)$ belongs to $D_{tsp}(T_{n,1}, i)$ by adjoining n and $n+1$.

Thus, $\{\{X \cup \{n+1\}$ if X ends with $n\} \cup \{X \cup \{n\}$ if X ends with $n-1\} \cup \{X \backslash \{n\} \cup \{n-2, n+1\}$ if X ends with n and X starts with $2\} \cup \{Y \cup \{n, n+1\}$ if Y starts with 1 and ends with $n-3\} \cup \{Y \cup \{n-1, n\}$ if Y ends with $n-4\} \cup \{Y \cup \{n-2, n+1\}$ if Y starts with 2 and ends with $n-3\} \cup \{Y \backslash \{n-3\} \cup \{n-1, n, n+1\}$ if Y starts with 3 ends with $n-3\}\} \subseteq D_{tsp}(T_{n,1}, i)$. (13)

Conversely, Suppose $Z \in D_{tsp}(T_{n,1}, i)$. Here all the elements of $D_{tsp}(T_{n,1}, i)$ ends with n or $n+1$.

If $n \in Z$ and $n+1 \notin Z$, then at least one vertex labeled $n-2$ or $n-1$.

- Suppose $n-1 \in Z, n-2 \in Z$, then $Z = X \cup \{n\}$, for some $X \in D_{tsp}(T_{n-1,1}, i-1)$.
- Suppose $n-1 \in Z, n-2 \notin Z$, then $Z = Y \cup \{n-1, n\}$, for some $Y \in D_{tsp}(T_{n-4,1}, i-2)$.

If $n+1 \in Z$, then at least one vertex labeled n or $n-1$ or $n-2$ or $n-3$ or $n-4$.

- Suppose $n \in Z, n-1 \in Z, n-2 \in Z$, then $Z = X \cup \{n+1\}$, for some $X \in D_{tsp}(T_{n-1,1}, i-1)$.
- Suppose $n \notin Z, n-2 \in Z, n-3 \in Z, n-4 \in Z$, then $Z \backslash \{n-2\} = X \cup \{n, n+1\}$, for some $X \in D_{tsp}(T_{n-1,1}, i-1)$.
- Suppose $n \notin Z, n-2 \in Z, n-3 \in Z, n-4 \notin Z$, then $Z = Y \cup \{n-2, n+1\}$, for some $Y \in D_{tsp}(T_{n-4,1}, i-2)$.
- Suppose Z starts with the vertex 1 and $n \in Z, n-3 \in Z, n-1 \notin Z, n-2 \notin Z$, then $Z = Y \cup \{n, n+1\}$, for some $Y \in D_{tsp}(T_{n-4,1}, i-2)$.
- Suppose Z starts with the vertex 1 and $n \in Z, n-1 \in Z, n-2 \notin Z, n-3 \notin Z$, then $Z = X \cup \{n+1\}$, for some $X \in D_{tsp}(T_{n-1,1}, i-1)$.
- Suppose Z starts with the vertex 3 and $n \in Z, n-1 \in Z, n-2 \notin Z, n-3 \notin Z$, then $Z \backslash \{n-1\} = Y \cup \{n-3, n, n+1\}$, for some $Y \in D_{tsp}(T_{n-4,1}, i-2)$.

Thus, $D_{tsp}(T_{n,1}, i) \subseteq \{\{X \cup \{n+1\}$ if X ends with $n\} \cup \{X \cup \{n\}$ if X ends with $n-1\} \cup \{X \backslash \{n\} \cup \{n-2, n+1\}$ if X ends with n and X starts with $2\} \cup \{Y \cup \{n, n+1\}$ if Y starts with 1 and ends with $n-3\} \cup \{Y \cup \{n-1, n\}$ if Y ends with $n-4\} \cup \{Y \cup \{n-2, n+1\}$ if Y starts with 2 and ends with $n-3\}$ $\{Y \backslash \{n-3\} \cup \{n-1, n, n+1\}$ if Y starts with 3 ends with $n-3\}\}$. (14)

From (13) and (14),

$D_{tsp}(T_{n,1}, i) = \{\{X \cup \{n+1\}$ if X ends with $n\} \cup \{X \cup \{n\}$ if X ends with $n-1\} \cup \{X \backslash \{n\} \cup \{n-2, n+1\}$ if X ends with n and X starts with $2\} \cup \{Y \cup \{n, n+1\}$ if Y starts with 1 and ends with $n-3\} \cup \{Y \cup \{n-1, n\}$ if Y ends with $n-4\} \cup \{Y \cup \{n-2, n+1\}$ if Y starts with 2 and ends with $n-3\} \cup \{Y \backslash \{n-3\} \cup \{n-1, n, n+1\}$ if Y starts with 3 ends with $n-3\}\}$, where $X \in D_{tsp}(T_{n-1,1}, i-1), Y \in D_{tsp}(T_{n-4,1}, i-2)$.

3. Conclusion

This paper discusses and analyses the twain secure perfect dominating sets of $T_{n,1}$. Using recursive method, we constructed the twain secure perfect dominating sets of $T_{n,1}$.

References

[1] S. Alikhani and Y.H. Peng (2009), *Dominating Sets and Domination Polynomials of Paths*, International Journal of Mathematics and Mathematical Sciences, vol. 2009, article ID 542040.

[2] F. Buckley and F. Harary (1990), *Distance in Graphs*, Addition-Wesley, Redwood City, California.

[3] E.J. Cockayne, P.J.P. Grobler, W.R. Grundlingh, J. Mungannga, and J.H. Van Vuuren (2005), *Protection of a Graph*, Utilitas Mathematica, vol. 67, pp. 19 − 32.

[4] K. Lal Gipson and S. Arun Williams (2021), *2−Edge Dominating Sets and 2−Edge Domination Polynomials of Paths*, Malaya Journal of Mathematik, vol. 9, pp. 844 − 849.

[5] K. Lal Gipson and S. Arun Williams, *2− Edge Dominating Sets and 2−Edge Domination Polynomials of tadpole ($T_{n,1}$)*, International virtual conference on recent trends and techniques in Mathematical and Computer Science, pp. 18-30.

[6] K. Lal Gipson and T. Subha (2022), *Secure Dominating Sets of Wheels*, IOSR Journal of Mathematics (IOSR-JM), vol. 18, no. 1, pp. 21 − 26.

[7] K. Lal Gipson and T. Subha (2020), *Secure Domination Polynomials of Paths*, International Journal of Scientific & Technology Research, vol. 9, no. 2, pp. 6517 − 6521.

[8] Vijayan and K. Lal Gipson (2012), Dominating Sets and Domination Polynomials in Square of Cycles, IOSR Journal of Mathematics (IOSR-JM), vol. 3, no. 4, pp. 04 − 14.

[9] Vijayan and K. Lal Gipson (2013), Dominating Sets and Domination Polynomials in Square of Paths, Open Journal in Discrete Mathematics, vol. 3, pp. 60 – 69.

[10] P.M. Weichsel (1994), *Dominating Sets in n-cubes*, Journal of Graph Theory, vol. 18, no. 5, pp. 479 − 488.