International Journal of
INTELLIGENT SYSTEMS AND APPLICATIONS IN ENGINEERING

ISSN:2147-6799 www.ijisae.org Original Research Paper

Detour Certified Domination Number of Degree Splitting Graphs of Certain Graphs

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Submitted: 02/11/2024 **Revised**: 17/12/2024 **Accepted**: 23/12/2024

Abstract: In this paper, we introduce the concept detour certified domination number of a graph. Also detour certified domination number of a certain classes of graphs are determined and some of its general properties are studied. A set S of vertices in a connected graph $G = (V, E)$ is called a detour set if every vertex not in S lies on a longest path between two vertices from S. A set D of vertices in G is called a dominating set of G if every vertex not in D has at least one neighbour in D. A set $T \subseteq V(G)$ is called a certified dominating set of G if T is a dominating set and every vertex in T has either zero or at least two neighbours in $V - T$. A set S is called a detour certified dominating set of G if S is both detour and certified dominating set of G. The detour certified domination number is the minimum cardinality of a detour certified dominating set in G.

Keywords*: Detour set, Dominating set, certified domination, Detour certified domination, Degree Splitting Graph*

AMS Subject classification: 05C12, 05C69.

1 Introduction

By a graph $G = (V, E)$ we mean a finite, connected, undirected graph with neither loops nor multiple edges. The order $|V|$ and size $|E|$ of G are denoted by p and q respectively. For graph theoretic terminology we refer to west [9]. A vertex v of G is said to be an extreme vertex if the subgraph induced by its neighbourhood is complete. The set of all extreme vertices is denoted by $Ext(G)$. For vertices x and y in a connected graph G, the detour distance $D(x, y)$ is the length of a longest $x - y$ path in G. An $x - y$ path of length $D(x, y)$ is called an $x - y$ detour. The closed interval $I_D[x, y]$ consists of all vertices lying on some $x - y$ detour of G. For $S \subseteq V, I_D[S] = \cup_{x,y \in S} I_D[x, y]$. A set S of vertices is a detour set if $I_D[S] = V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. The concept of detour set in graph was introduced by G.Chartrand et.all [2,3]. For distance related terminology we refer to Buckley and Harary [1].

A set $S \subseteq V(G)$ in a graph G is a dominating set of G if every vertex v in $V - S$, there exists a vertex $u \in S$ such that v is adjacent to u. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G [5].

A vertex of degree 0 is called an isolated vertex and a vertex of degree 1 is called an end vertex or a pendant vertex. A vertex that is adjacent to a pendant vertex is called a support vertex. A vertex of degree $p - 1$ is called a full vertex. The concept of certified domination in graph was introduced by Magoba Dettlaff et.all [4].

Definition 1.1 Let $G = (V, E)$ be any graph of order n. A subset $S \subseteq V(G)$ is said to be a certified dominating set of G if S is a dominating set of G and every vertex in S has either zero or at least two

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neighbours in $V - S$. The certified domination number denoted by $\gamma_{cer}(G)$ is the minimum cardinality of certified dominating sets in G.

For graph G, some subsets of V are detour set but not certified dominating set in G. Further, some other are certified dominating set but not detour sets in G. Moreover, some of them that are both certified dominating and detour set in G.

Here we see that a detour set is not in general a certified dominating set in G. The converse also not true in general. This motivated to study a new domination concept of detour certified domination. Hence, we study that subsets of $V(G)$ that are both a certified dominating and a detour set in G. We call these sets as detour certified domination number as the minimum cardinality taken over all the detour certified dominating sets of G.

Definition 1.2 Let $G = (V, E)$ be any connected graph with at least two vertices. A subset $S \subseteq V(G)$ is called a detour certified dominating set of G if S is both detour and certified dominating set of G. A detour certified dominating set S is said to be a minimum detour certified dominating set of G if there exists no detour certified dominating set S' of G such that $|S'| < |S|$. The cardinality of a minimum detour certified dominating set of G is the detour certified dominating number of G. It is denoted by $c\gamma_d(G)$.

Any detour certified dominating set S of cardinality $c\gamma_d(G)$ is called a $c\gamma_d -$ set of G or $c\gamma_d(G) -$ set.

2. Some basic results

In this section, we call some definitions and basic results of detour set and certified dominating set of a graph which will be used throughout the paper.

Theorem 2.1 [3] *Every end vertex of G belongs to every detour set of G.*

Theorem 2.2 [4] *Every support vertex of G belongs to every certified dominating set of G.*

3. Detour Certified Domination number in degree splitting graphs.

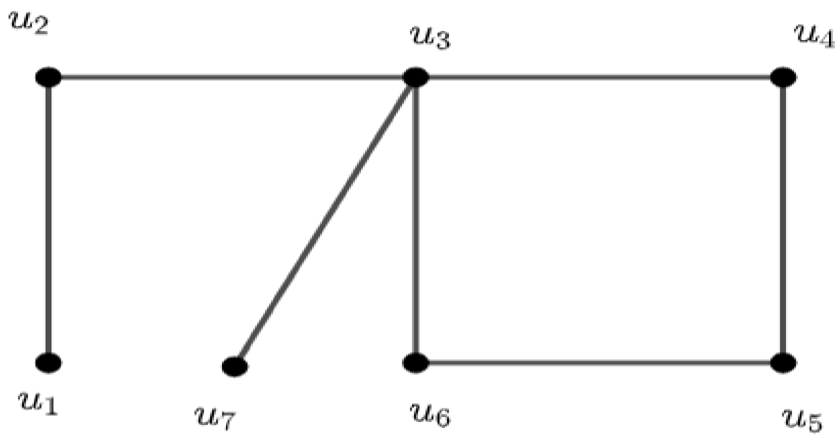
In this section, we define the detour certified domination number for degree splitting graphs some families of graphs.

Definition 3.1

Let $G = (V, E)$ be a connected graph of order $p \geq 2$ with $V(G) = S_1 \cup S_2 \cup \ldots \cup S_r \cup R$, where S_i is the set having at least two vertices of same degree and $R = V(G) \cup S_i$ for $1 \leq i \leq r$. The degree splitting graph $DS(G)$ is obtained from G by adding vertices $v_1, v_2, \ldots, v_n$ and joining v_i to each vertices in S_i for $1 \leq i \leq r$.

Example 3.2

Consider a connected graph G and their corresponding degree splitting graph $DS(G)$ given in Figure 3.1



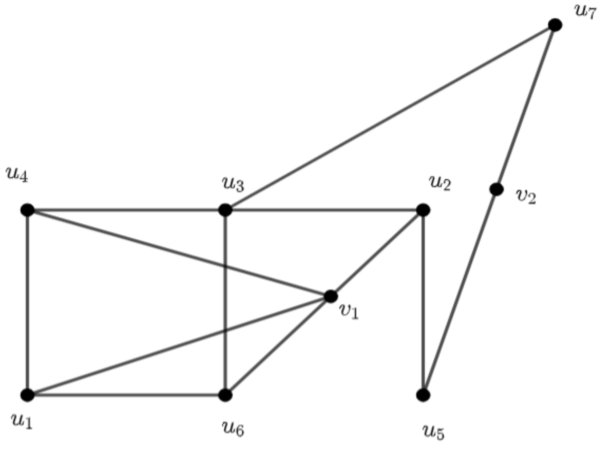


Figure 3.1

Here $S_1 = \{u_2, u_4, u_5, u_6\}, S_2 = \{u_1, u_7\}$ and $R = \{u_3\}$.

Remark 3.3

If $V(G) = \cup S_i, 1 \leq i \leq r$ for any connected graph G, then $R = \phi$ in the degree splitting graph.

Remark 3.4

The graph G given in Figure 3.1(a) and the graph $DS(G)$ given in Figure 3.1, it is clear that $S = \{u_1, u_2, u_3, u_5, u_7\}$ and $S_1 = \{u_1, u_3, u_7, v_1\}$ are the minimum detour certified dominating set of G and $DS(G)$, respectively. Thus $c\gamma_d(G) = 5$ and $c\gamma_d\big(DS(G)\big) = 4$.

So, the detour certified domination number of G and $DS(G)$ are different.

Observation 3.5

For any connected graph G, there is no obvious relation connecting $c\gamma_d(G)$ and $c\gamma_d\big(DS(G)\big)$.

Example for $c\gamma_d\big(DS(G)\big) < c\gamma_d(G)$.

Let us consider the connected graph G in Figure 3.2.

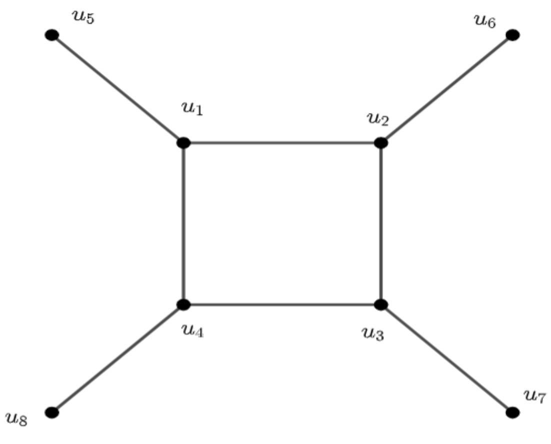
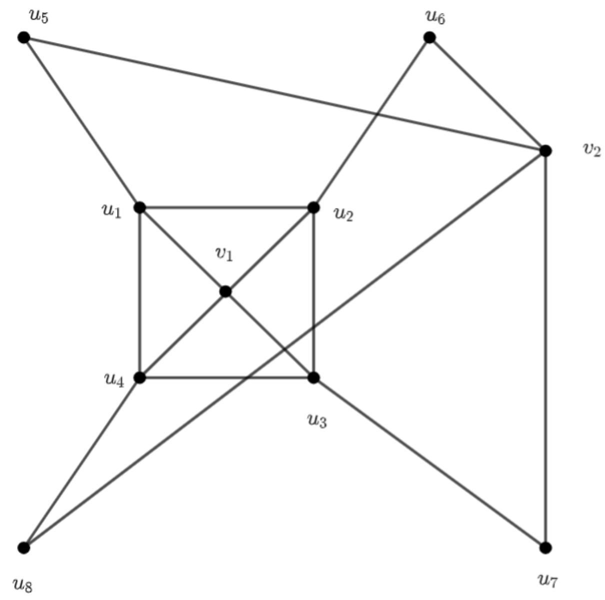


Figure 3.2

Here $S = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ is the unique detour certified dominating set of G and hence, $c\gamma_d(G) = |S| = 8$.

Now, we degree split each vertex of G, to obtained a new connected graph $DS(G)$ and is given in Figure 3.3.



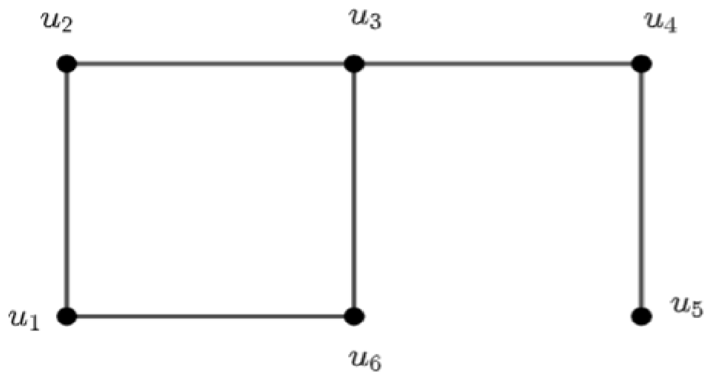
$DS(G)$

Figure 3.3

Here $S_1 = \{v_1, v_2\}$ is a minimum detour certified dominating set of $DS(G)$ and hence $c\gamma_d(DS(G)) = |S_1| = 2$.

In this case, $c\gamma_d(DS(G)) < c\gamma_d(G)$.

Consider the connected graph G given in Figure 3.4.



G

Figure 3.4

Here $S = \{u_1, u_4, u_5\}$ is a minimum detour certified dominating set of G and hence, $c\gamma_d(G) = |S| = 3$.

Now, we degree split each vertex of G, to obtained a new connected graph $DS(G)$ and is given in Figure 3.5.

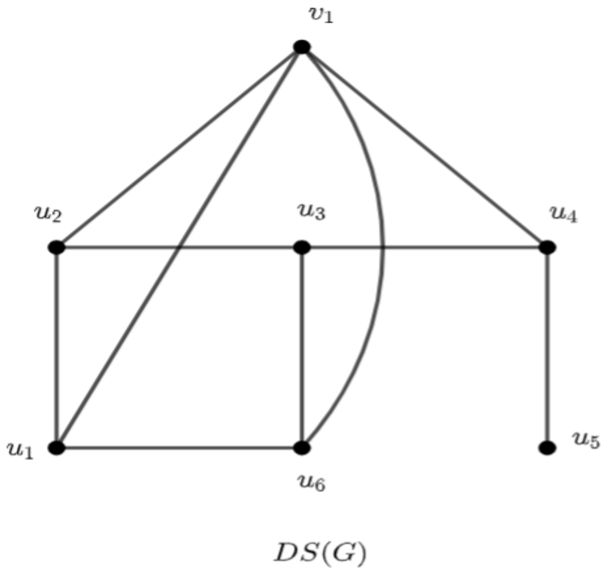


Figure 3.5

Here $S_1 = \{v_1, u_4, u_5\}$ is a minimum detour certified dominating set of $DS(G)$ and hence, $c\gamma_d\big(DS(G)\big) = |S_1| = 3$.

Thus, in this case, $c\gamma_d\big(DS(G)\big) = c\gamma_d(G)$.

Example for $c\gamma_d\big(DS(G)\big) > c\gamma_d(G)$.

Consider the connected graph G, given in Figure 3.6.

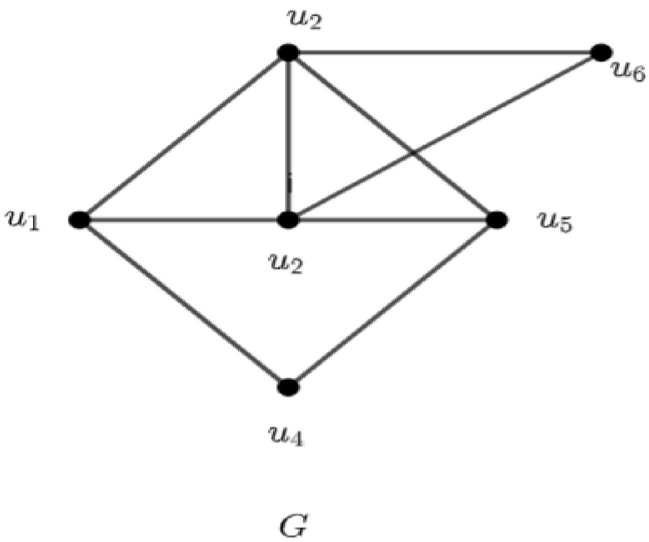
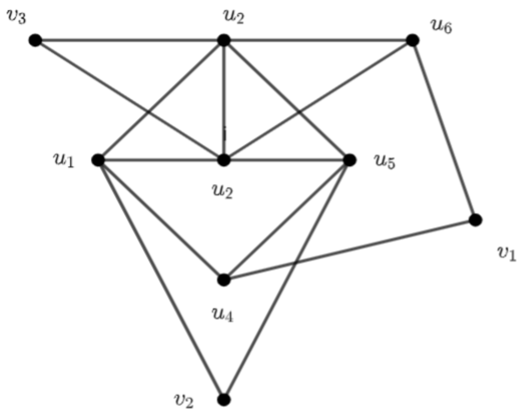


Figure 3.6

Here $S = \{u_3, u_5\}$ is a minimum detour certified dominating set of G and hence $c\gamma_d(G) = |S| = 2$.

Now, we degree split each vertex of G, to obtained a new connected graph $DS(G)$ and it is given in Figure 3.7.



$DS(G)$

Figure 3.7

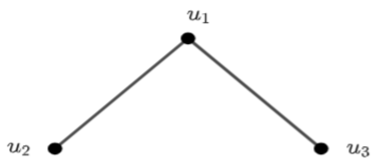
Here $S_1 = \{u_3, u_4, u_5\}$ is a minimum detour certified dominating set of $DS(G)$ and hence, $c\gamma_d\big(DS(G)\big) = |S_1| = 3$.

Thus, in this case $c\gamma_d\big(DS(G)\big) > c\gamma_d(G)$.

Remark 3.6

Every end vertex of G and every support vertex of G need not belong to some detour certified dominating set of $DS(G)$.

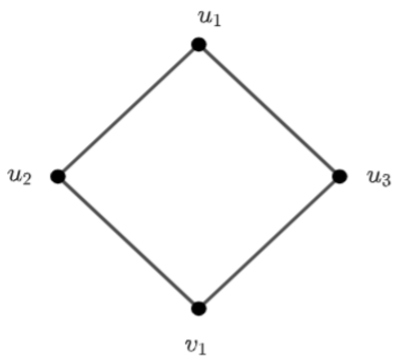
For example, consider the graph given in Figure 3.8.



G

Figure 3.8

The degree splitting graph $DS(G)$ of G is given in Figure 3.9.



$DS(G)$

Figure 3.9

Here $S = \{u_1, v_1\}$ is a detour certified dominating set of $DS(G)$. But u_2 and u_3 are the end vertices of G does not in S.

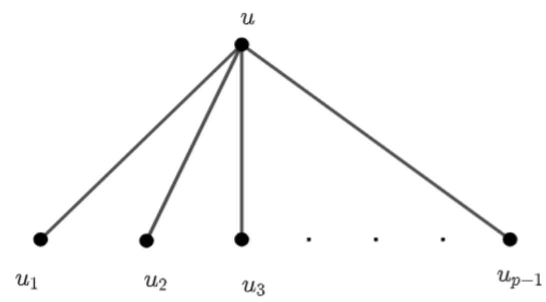
Similarly, if we consider $S = \{u_1, u_3\}$, then S is a detour certified dominating set of $DS(G)$. But here u_1 is a support vertex of G, does not in S.

Theorem 3.7

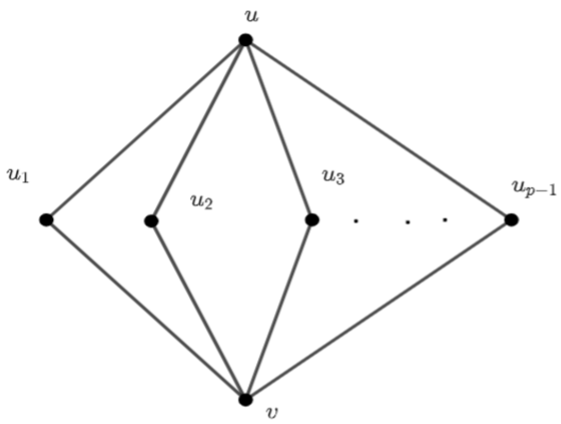
For integer $p \geq 2, c\gamma_d\left(DS(K_{1,p-1})\right) = 2$.

Proof

Let $u_1, u_2, \dots, u_{p-1}$ be the end vertices and u be the central vertex of the star graph $K_{1,p-1}$. Let v be the corresponding vertex which is added to obtained that graph $DS(K_{1,p-1})$. So $V\left(DS(K_{1,p-1})\right) = \{u, u_1, u_2, \dots, u_{p-1}, v\}$ and hence $\left|V\left(DS(K_{1,p-1})\right)\right| = p + 1$.



$K_{1,p-1}$



$DS(K_{1,p-1})$

Figure 3.10

Since $DS(K_{1,p-1})$ is connected, that $c\gamma_d\left(DS(K_{1,p-1})\right) \geq 2$. Consider $S = \{u, u_i\}$ or $S = \{v, u_i\}$ for $1 \leq i \leq p - 1$. Then clearly every vertices $V\left(DS(K_{1,p-1})\right)\backslash S$ lies on a detour path between vertices form S. Also vertices in S dominates every vertices of $V\left(DS(K_{1,p-1})\right)$. Therefore that S is not a detour certified dominating set of $DS(K_{1,p-1})$. Now, we consider $S_1 = \{u, v\}$. Clearly u_i for $i \leq i \leq p - 1$ is dominated by u and v. Since $p \geq 3$, every vertices in S_1 has at least two neighbours in $V\left(DS(K_{1,p-1})\right)\backslash S_1$. Therefore that S_1 is a certified dominating set $DS(K_{1,p-1})$. Moreover, every vertices in $V\left(DS(K_{1,p-1})\right)\backslash S$, lies on the detour path between vertices from u and v. So that S_1 itself is a detour certified dominating set of $DS(K_{1,p-1})$. Thus, $c\gamma_d\left(DS(K_{1,p-1})\right) \leq |S_1| = 2$. Hence, we conclude $c\gamma_d\left(DS(K_{1,p-1})\right) = 2$.

Example 3.8

Consider $DS(K_{1,7})$ given in the Figure 3.11, by Theorem 3.7, $S = \{u, v\}$ is a minimum detour certified dominating set of $DS(K_{1,7})$ and hence, $c\gamma_d\left(DS(K_{1,7})\right) = |S| = 2$.

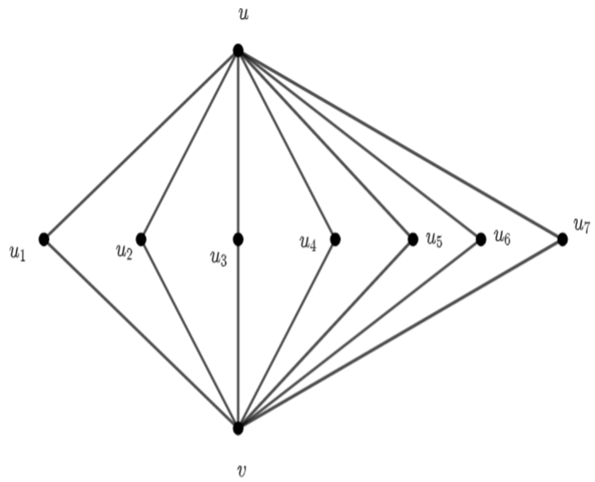


Figure 3.11: $DS(K_{1,7})$

Corollary 3.9

For the star graph $K_{1,p-1}$, $c\gamma_d(K_{1,p-1}) = c\gamma_d\left(DS(K_{1,p-1})\right) - p - 2$.

Theorem 3.10

Let G be a graph obtained from $DS(K_{1,p-1})$ by joining pendant vertices to each of $p-1$ vertices, $p \geq 3$. Then $c\gamma_d(DS(G)) = 2p - 2$.

Proof

Let $u_1, u_2, \ldots, u_{p-1}$ be the end vertices and u be the central vertex of the star graph $K_{1,p-1}$. Let v be the corresponding vertex, which is added to obtained the graph $DS(K_{1,p-1})$. Now, let G be the graph obtained from the $DS(K_{1,p-1})$ by joining the pendant vertices $v_1, v_2, \ldots, v_{p-1}$ to each of $u_1, u_2, \ldots, u_{p-1}$, respectively and the graph is given in Figure 3.12.

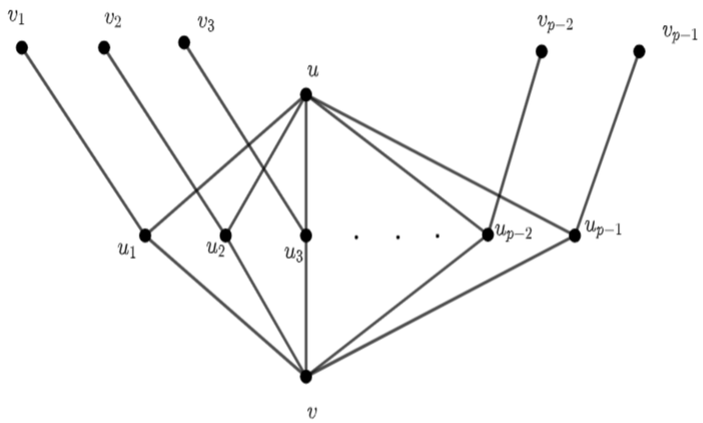


Figure: 3.12: G

Here $S = \{u_1, u_2, \ldots, u_{p-1}, v_1, v_2, \ldots, v_{p-1}\}$ be the set if support vertices and end vertices of G. So $c\gamma_d(G) \geq |S| = 2p - 2$. Now, it is clear that S itself from a detour certified dominating set of G and so $c\gamma_d(G) \leq |S| = 2p - 2$. Hence, $c\gamma_d(G) = 2p - 2$.

Example 3.11

Consider the graph G obtained from $DS(K_{1,7})$ by adding one pendant vertices to each of 7 vertices of degree 2. Then, $c\gamma_d(G) = 14$.

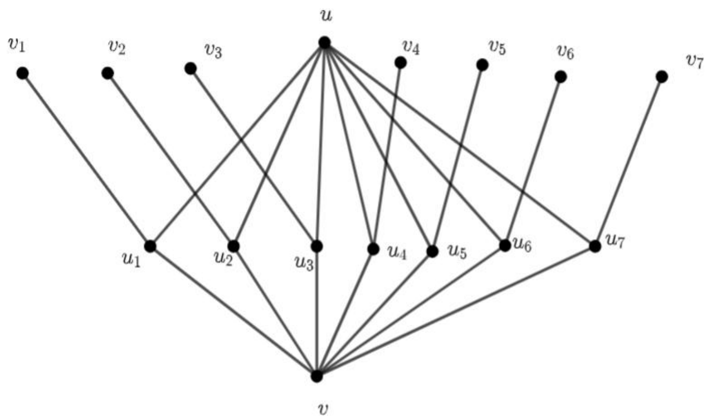


Figure 3.13: G

By Theorem 3.10, $S = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a minimum detour certified dominating set of G and hence, $c\gamma_d(G) = |S| = 14$.

Theorem 3.12

For the bistar graph $B_{m,n}(m \geq n \geq 1)$,

$$c\gamma_d\left(DS(B_{m,n})\right) = \begin{cases} 2 \; if \; m = n \\ 3 \; if \; m \neq n \; and \; n \geq 2 \\ 4 \; if \; n = 1 \; and \; m > 1 \end{cases}.$$

Proof

Consider our bistar graph $B_{m,n}$ with $V(B_{m,n}) = \{u, v, u_i, v_j; 1 \leq i \leq m, 1 \leq j \leq n\}$. Here that u_i and v_j are the vertices adjacent with u and v, repetively. To obtain $DS(B_{m,n})$, three cases where axis.

Case (i) $m = n$

Let x_1 and x_2 be the corresponding vertices which are added to obtain $DS(B_{m,n})$. Then, $V\left(DS(B_{m,n})\right) = \{u, v, u_i, v_i, x_1, x_2; 1 \leq i \leq m\}$. Thus, $\left|V\left(DS(B_{m,n})\right)\right| = m + n + 4$. This graph is given in Figure 3.14.

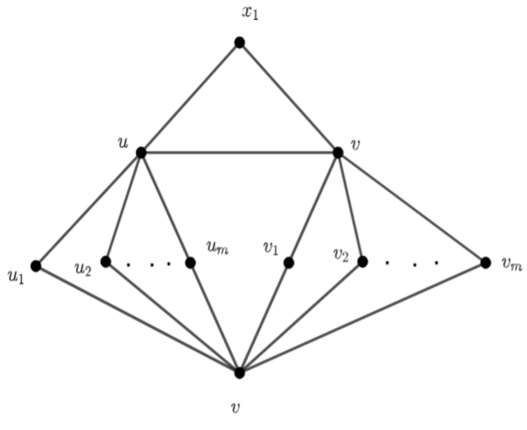


Figure 3.14: $DS(B_{m,n})$

Since $DS(B_{m,n})$ is connected that $c\gamma_d\left(DS(B_{m,n})\right) \geq 2$. Consider $S = \{x_1, u\}$ or $\{x_1, v\}$. Then clearly every vertices in $V\left(DS(B_{m,n})\right) \setminus S$ lies on a detour path between vertices from S. So that S is a detour set of $DS(B_{m,n})$. But that $x_2, u_1, u_2, \dots, u_m$ or $x_2, v_1, v_2, \dots, v_m$ is not dominated by any vertices from $S = \{x_1, v\}$ or $S = \{x_1, u\}$, respectively. So that S is not a detour certified dominating set of $DS(B_{m,n})$. Now, consider $S_1 = \{x_1, x_2\}$. Since u, v is dominated by x_1 and $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m$ are dominated by x_2, that S_1 dominates $V\left(DS(B_{m,n})\right)$. Since $m, n \geq 1$, every vertices in S_1 has at least two neighbours in $V\left(DS(B_{m,n})\right) \setminus S_1$. Therefore, that S_1 is a certified dominating set of $DS(B_{m,n})$. Moreover, every vertices in

$V\left(DS(B_{m,n})\right)\backslash S_1$ lies on the detour path between vertices from S_1. So that S_1 itself is a detour certified dominating set of $DS(B_{m,n})$ and so $c\gamma_d\left(DS(B_{m,n})\right) \le |S_1| = 2$. Hence we conclude, $c\gamma_d\left(DS(B_{m,n})\right) = 2$.

Case (ii) $m \ne n$ and $n \ge 2$

Let x_1 be the corresponding vertex added to $B_{m,n}$ to obtained the graph $DS(B_{m,n})$. Then $V\left(DS(B_{m,n})\right) = \{u, v, u_i, v_j, x_1; 1 \le i \le m, 1 \le j \le n\}$. Thus $\left|V\left(DS(B_{m,n})\right)\right| = m + n + 3$. This graph contains no support and end vertices and is shown in Figure 3.15.

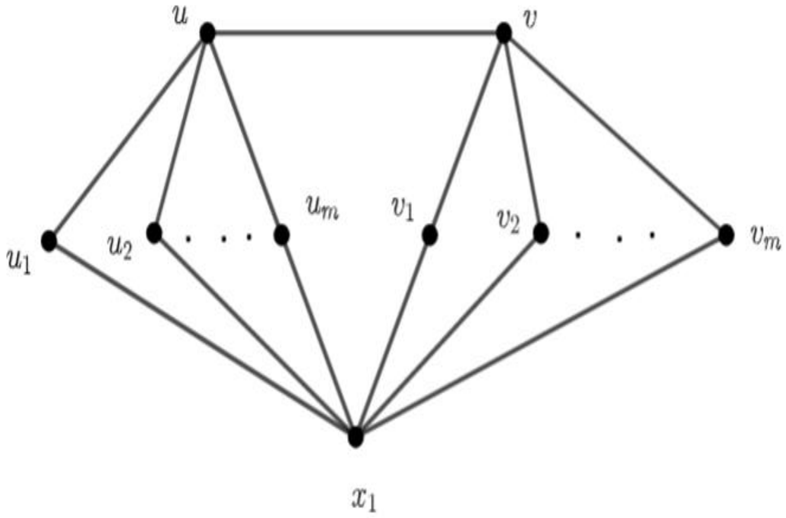


Figure 3.15: $DS(B_{m,n})$

Since $DS(B_{m,n})$ is connected, that $c\gamma_d\left(DS(B_{m,n})\right) \ge 2$. Consider, either $S = \{u, x_1\}$ or $S = \{v, x_1\}$. If $S = \{u, x_1\}$, then u dominates $v, u_1, u_2, \dots, u_m$ and x_1 dominates $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m$. Therefore, S is a dominating set of $DS(B_{m,n})$. Since $m, n \ge 2$, every vertices in S has at least two neighbours in $V\left(DS(B_{m,n})\right)\backslash S$. Therefore, that S is a certified dominating set of $DS(B_{m,n})$. Now, it is clear that every longest path of u, x_1. Consider $v, v_1, v_2, \dots, v_n$ as the internal vertices. But it does not contain $u_1, u_2, \dots, u_m$ as the detour path between vertices from S. So that S is not a detour certified dominating set of $DS(B_{m,n})$. As similar we discuss above, if $S = \{v, x_1\}$ then $v_1, v_2, \dots, v_n$ does not lie on any detour path between vertices from S. So $S = \{u, x_1\}$ or $\{v, x_1\}$ is not a detour certified dominating set of $DS(B_{m,n})$. Also there does not exist a detour certified dominating set of $DS(B_{m,n})$ with two vertices. Thus, $c\gamma_d\left(DS(B_{m,n})\right) \ge 3$. Now consider $S_1 = \{u, v, x_1\}$. Since $S \subseteq S_1$, that S_1 is a certified dominating set of $DS(B_{m,n})$. Moreover, $u - v$ detour path contains all the vertices of $DS(B_m, n)\backslash S_1$ and x_1 as the internal vertices. Therefore that S_1 is a detour certified dominating set of $DS(B_{m,n})$ and hence, $c\gamma_d\left(DS(B_{m,n})\right) \le |S_1| = 3$. Thus, $c\gamma_d\left(DS(B_{m,n})\right) = 3$.

Case (ii) $n = 1$ and $m > 1$

In this case, the $B_{m,1}, v$ is adjacent with exactly one end vertex v_1 and u is adjacent with more than two end vertices. Let x_1 be the degree splitting vertex corresponding to which is added to $V(B_{m,1})$ to obtain $DS(B_{m,1})$. Then $V\left(DS(B_{m,1})\right) = \{u, v, u_i, v_1; x_1; 1 \le i \le m\}$ and thus, $\left|V\left(DS(B_{m,1})\right)\right| = m + 4$. This graph is shown in Figure 3.16.

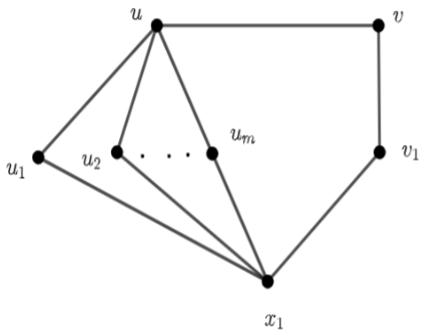


Figure 3.16: $DS(B_{m,1})$.

Here $S = \{u, x_1\}$ or $S = \{v, x_1\}$ dominates every other vertices of $DS(B_{m,1})$. Since u is adjacent with $v, u_1, u_2, \dots, u_m$ and v is adjacent with u, v_1 and x_1 is adjacent with $u_1, u_2, \dots, u_m, v_1$, that S is a certified dominating set of $DS(B_{m,1})$. As similar discussion discussed in case(ii), that S is not a detour certified dominating set of $DS(B_{m,1})$. Now consider $S_1 = \{u, v, x_1\}$. It is clear that S_1 is a detour set of $DS(B_{m,1})$. But v has exactly one neighbour v_1 in $V\left(DS(B_{m,1})\right)\backslash S_1$. Therefore that S_1 itself is not a detour certified dominating set of $DS(B_{m,1})$. If we rearrange S_1 as $S_1 = \{u, v_1, x_1\}$, then v_1 has exactly one neighbour in $V\left(DS(B_{m,1})\right)\backslash S_1$. Also if we consider any vertex from $u_1, u_2, \dots, u_m$ then that u_i has exactly one neighbour in $V\left(DS(B_{m,1})\right)\backslash S_1$. So $c\gamma_d\left(DS(B_{m,1})\right) \geq 4$. Let us consider $S_2 = \{u, v, v_1, x_1\}$. Then it is clear that S_2 is both detour and certified dominating set of $DS(B_{m,1})$ and so $c\gamma_d\left(DS(B_{m,n})\right) \leq |S_2| = 4$. Hence $c\gamma_d\left(DS(B_{m,n})\right) = 4$.

Thus $c\gamma_d\left(DS(B_{m,n})\right) = \begin{cases} 2 \text{ } if \text{ } m = n \\ 3 \text{ } if \text{ } m \neq n \text{ } and \text{ } n \geq 2 \\ 4 \text{ } if \text{ } n = 1 \text{ } and \text{ } m > 1 \end{cases}$.

Example 3.13

Consider the graph of $DS(B_{4,4})$ given in Figure 3.17.

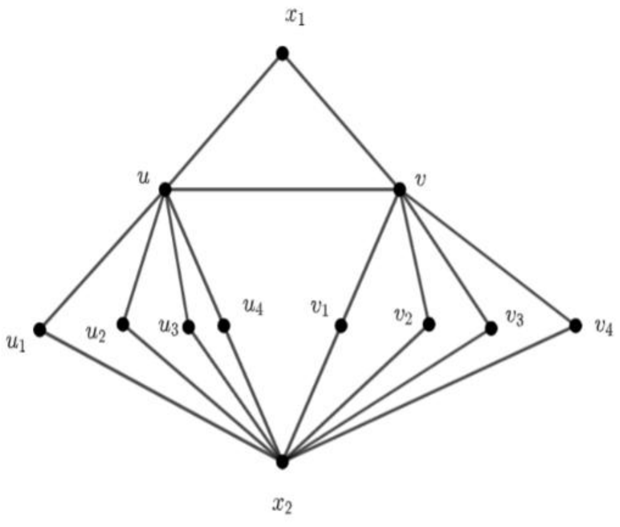


Figure 3.17: $DS(B_{4,4})$

By Theorem 3.12 case(i) $S = \{x_1, x_2\}$ is a minimum detour certified dominating set of $DS(B_{4,4})$ and hence $c\gamma_d\left(DS(B_{4,4})\right) = |S| = 2$.

Consider the graph of $DS(B_{5,4})$ given in Figure 3.18.

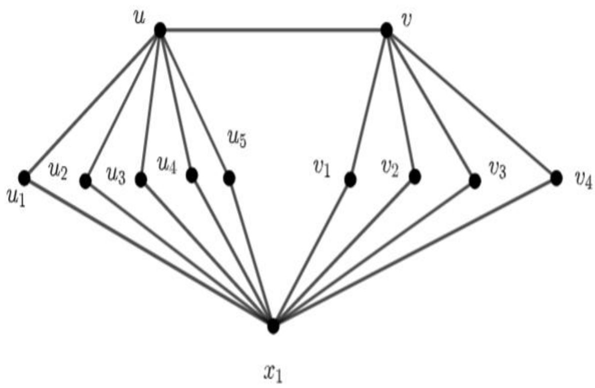


Figure 3.18: $DS(B_{5,4})$

By Theorem 3.12, case(ii) $S_1 = \{u, v, x_1\}$ is a minimum detour certified dominating set of $DS(B_{5,4})$ and hence, $c\gamma_d\left(DS(B_{5,4})\right) = |S_1| = 3$.

Consider the graph of $DS(B_{5,1})$ given in Figure 3.19.

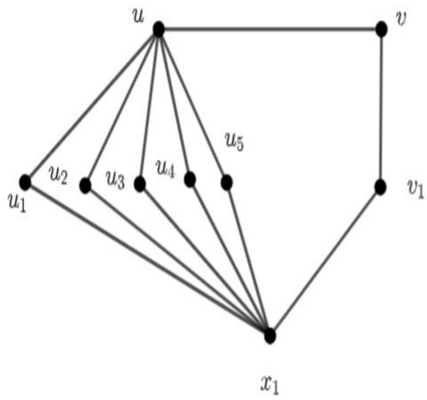


Figure 3.19:$DS(B_{5,1})$

By Theorem 3.12, case(iii), $S_2 = \{u, v, v_1, x_1\}$ is a minimum detour certified dominating set of $DS(B_{5,1})$ and hence, $c\gamma_d\left(DS(B_{5,1})\right) = |S_2| = 4$.

Corollary 3.14

For the bistar graph $B_{m,n}(m \geq 1)$, $c\gamma_d\left(DS(B_{m,n})\right) - c\gamma_d\left(DS(B_{m,n})\right) = 2m$.

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